TIGHT BINDING BOOK

# ALGEBRAIC EQUATIONS

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## AN INTRODUCTION TO THE THEORIES OF LAGRANGE AND GALOIS

BY

EDGAR DEHN, PH.D.

*Instructor of Mathematics in Columbia University*

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# LIST OF THEOREMS

## LIST OF ADOPTED CONVENTIONS

While it is neither possible nor desirable to stereotype algebraic notation, certain conventions are expedient if waived when inopportune.

| | |
|---|---|
| $\sum$ | sum of functions |
| $\prod$ | product of functions |
| $x$ | argument of function |
| $x_i,\alpha_i$ | roots of function |
| $f,g$ | function of $x$ |
| $\varphi,\psi$ | function of the $x_i$ |
| $\varphi(x_i)_1^n,\varphi(x_i)$ | $= \varphi(x_1, \ . \ . \ . \ , x_n)$ |
| $\psi_i$ | conjugate functions |
| $F$ | reducible function |
| $f$ | irreducible function |
| $R,r$ | rational function* |
| $j$ | integral function |
| $c$ | cyclotomic function |
| $S$ | symmetric function* of the $x_i$ |
| $s$ | symmetric sum* in the $x_i$ |
| $s^{(n)}$ | same on $n$ letters $x_i$ |
| $A$ | alternating function of the $x_i$ |
| $R$ | resultant |
| $D$ | discriminant in homogeneous form |
| $\Delta$ | discriminant in non-homogeneous form |
| $G_2,G_3$ | invariants of cubic |
| $g_2,g_3$ | invariants of biquadratic |
| $W$ | weight of function |
| $D$ | total degree of function |
| $v$ | elementary Galoisian function |
| $g(v)$ | Galoisian function |
| $G(v)$ | complete Galoisian function |

* Symbols for symmetric and rational functions may be used without reference to a definite form of those functions.

| | |
|---|---|
| $g(v)$ | primary Galoisian function |
| $g(v) = 0$ | Galoisian resolvent |
| $r(\psi) = 0$ | ordinary resolvent |
| $(\epsilon,\psi)$ | Lagrange's solvent |
| $s,t$ | permutation (substitution) on the $x_i$ |
| 1 | identical permutation |
| $(x_1x_i)_2^n$ | $= (x_1x_2), \ldots, (x_1x_n)$ |
| $G$ | group of permutations on the $x_i$ |
| $\{G\}$ | explicit notation of groups |
| $\varphi$ | function belonging to $G$ |
| $S$ | symmetric group |
| $A$ | alternating group |
| 1 | identical group |
| $v$ | function belonging to 1 |
| $H$ | subgroup of $G$ |
| $\psi$ | function belonging to $H$ |
| $\xi$ | function belonging to group that contains $H$ but no other permutation of $G$ |
| $\Xi$ | group of $\xi$ |
| $t$ | permutation in $G$ but not in $H$ |
| $H_t$ | conjugate subgroup |
| $Ht$ | partition of $G$ |
| $D$ | greatest common subgroup |
| $N, J$ | normal (invariant) subgroup |
| $\bar{N}$ | maximum normal subgroup |
| $X_i$ | group of $x_i$ |
| $Z$ | central subgroup |
| $s$ | normal permutation |
| $V$ | quadratic group |
| $M$ | metacyclic group |
| $Q$ | quotient of permutation-groups |
| $q$ | permutation in $Q$ |
| $\{s,t\}$ | group generated by $s$ and $t$ |
| $C,\{s\}$ | cyclic group $s,s^2, \ldots$ |
| $\{s_i\}_1^n,\{s_i\}$ | group $s_1,s_2, \ldots, s_n$ |
| $\langle G\rangle,G$ | Galoisian group |
| $\langle A\rangle,A$ | Abelian group |
| $C$ | class of permutations |

| | |
|---|---|
| $c$ | permutation in $C$ |
| $\sigma,\tau$ | permutation on the $\psi_i$ |
| $\Gamma$ | group of such permutations |
| $\sigma$ | substiiution in domain |
| $\langle\Gamma\rangle$ | substitution-group $= \langle G\rangle$ |
| $G/N$ | abstract quotient or factor-group |
| $a,b$ | element |
| $1$ | identical element |
| $r,\rho$ | order of group or permutation |
| $r_g,g$ | order of group $G$ |
| $G_r$ | group of order $r$ |
| $n$ | degree of group or permutation |
| $G^n$ | group of degree $n$ |
| $r_s$ | order of permutation $s$ |
| $\rho$ | relative order of permutation |
| $j$ | index of group |
| $j_g$ | index of $G$ in $S$ |
| $j_{hg}$ | index of $H$ in $G$ |
| $\Omega$ | domain |
| $(a)$ | $= \Omega(a)$ |
| $(a_i)_1^n,(a_i)$ | $= (a_1, \ . \ . \ . \ , a_n)$ |
| $\omega$ | number in $\Omega$ |
| $\theta$ | primitive number in $\Omega$ |
| $p$ | prime number |
| $g$ | primitive root of $p$ |
| $r$ | rational number |
| $\epsilon$ | primitive root of unity |
| $\omega$ | primitive cube root of unity |
| $i$ | primitive fourth root of unity |

$\varphi(n)$ number of positive integers prime to $n$ and smaller than $n$

# CHAPTER I

# INTEGRAL FUNCTION

## §1. INTERPOLATION

It is the test of a perfect theory that its study leaves in the mind of the student a sense of accomplishment and beauty. One such theory is the theory of algebraic equations created by two men of genius: Lagrange and Galois.

The solution of equations is the principal object of **algebra,** as the original meaning of this term implies, but it is understood that the means of an **algebraic solution** are confined to **algebraic operations** alone which, beside the rational operations of arithmetic, include the extraction of roots.[1]

It will be noticed that equations in one unknown only are treated here and that the problem of elimination is not under discussion.

An algebraic equation is of the form

$$F(x) = 0$$

where $F(x)$ is an algebraic function. As single-valued algebraic functions are necessarily rational,[2] we may set

$$F(x) = \frac{f(x)}{g(x)}$$

with integral $f(x)$ and $g(x)$ and proceed to an equation

$$f(x) = 0$$

where $f(x)$ is an integral rational function.

[1] By such operations we mean here nothing more than the use of the proper signs, by extraction of roots nothing more than the use of the radical sign.

[2] Cf. Townsend, Functions of a Complex Variable, Art. 54, p. 290.

Hence we begin by noting some properties of integral rational functions,[1] or polynomials. Every such function of the variable $x$ can be respresented in the form

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n.$$

So to determine the $n + 1$ coefficients $a_i$ that for $n + 1$ given values

$$x_0, x_1, \ldots, x_n$$

of the variable the function assumes $n + 1$ given values

$$y_0, y_1, \ldots, y_n$$

is a problem of interpolation. It is solved by the

**Interpolation formula of Lagrange:**

$$\begin{aligned} f(x) = {} & y_0\frac{(x - x_1)(x - x_2)(x - x_3) \ldots (x - x_n)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3) \ldots (x_0 - x_n)} \\ & + y_1\frac{(x - x_0)(x - x_2)(x - x_3) \ldots (x - x_n)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3) \ldots (x_1 - x_n)} \\ & + y_2\frac{(x - x_0)(x - x_1)(x - x_3) \ldots (x - x_n)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3) \ldots (x_2 - x_n)} \\ & \quad \ldots\ldots\ldots \\ & + y_n\frac{(x - x_0)(x - x_1) \ldots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \ldots (x_n - x_{n-1})}. \end{aligned}$$

Setting

$$g(x) = (x - x_0)(x - x_1)(x - x_2) \ldots (x - x_n).$$

we have as its derivative

$$\begin{aligned} g'(x) = {} & (x - x_1)(x - x_2) \ldots (x - x_n) \\ & + (x - x_0)(x - x_2) \ldots (x - x_n) \\ & \quad \ldots\ldots\ldots \\ & + (x - x_0)(x - x_1) \ldots (x - x_{n-1}), \end{aligned}$$

whence

$$\begin{aligned} g'(x_0) &= (x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n) \\ g'(x_1) &= (x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n) \\ & \quad \ldots\ldots\ldots \\ g'(x_n) &= (x_n - x_0)(x_n - x_1) \ldots (x_n - x_{n-1}). \end{aligned}$$

[1] If no confusion can arise, we shall simply call them integral functions.

Consequently, the interpolation formula can be written

$$f(x) = g(x)\left[y_0\frac{1}{g'(x_0)(x - x_0)} + y_1\frac{1}{g'(x_1)(x - x_1)} + \dots\right]$$

or

$$f(x) = g(x)\sum_{i=0}^{n}\frac{y_i}{g'(x_i)(x - x_i)}.$$

## §2. DIVISION

If

$$f = a_0x^n + a_1x^{n-1} + \dots + a_n$$

and

$$\varphi = \alpha_0x^\nu + \alpha_1x^{\nu-1} + \dots + \alpha_\nu$$

are two integral functions of $x$ such that

$$n \geqq \nu,$$

we form the integral function

$$f_1 = f - \frac{a_0}{\alpha_0}x^{n-\nu}\,.\,\varphi$$

which is of degree smaller than $n$. Then setting

$$f_1 = b_0x^m + b_1x^{m-1} + \dots + b_m,$$

we form the integral function

$$f_2 = f_1 - \frac{b_0}{\alpha_0}x^{m-\nu}\,.\,\varphi$$

which is of degree smaller than $m$. So we continue until we reach the integral function

$$f_k = f_{k-1} - \frac{p_0}{\alpha_0}x^{h-\nu}\,.\,\varphi$$

which is of degree smaller than $\nu$ and then adding obtain

$$f = \frac{a_0x^{n-\nu} + b_0x^{m-\nu} + \dots + p_0x^{h-\nu}}{\alpha_0}\varphi + f_k$$

or

$$f = q\,.\,\varphi + r.$$

Since such an operation is ordinary division of two integral functions $f$ and $\varphi$ giving $q$ as quotient and $r$ as remainder, we can note that

(1) **quotient and remainder in the division of two integral functions are themselves integral in the variable of those functions,**

and also in their coefficients when $\alpha_0 = 1$:

$$q = j_1(x\,|a_i|\,\alpha_i)$$
$$r = j_2(x\,|a_i|\,\alpha_i).$$

It appears now that **Euclid's algorism** for finding the greatest common factor of two integers can be applied to two integral functions $f$ and $f_1$ of $x$. For we have

$$f = q_1 f_1 + f_2$$
$$f_1 = q_2 f_2 + f_3$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$f_{r-3} = q_{r-2} f_{r-2} + f_{r-1}$$
$$f_{r-2} = q_{r-1} f_{r-1} + f_r,$$

which terminates since the degrees of the $f_i$ constantly decrease until $f_r$ becomes independent of $x$. If $f$ and $f_1$ have a common factor other than a constant, this factor is evidently contained in every following $f_i$ and

$$f_r = 0.$$

But then, by the last equation of Euclid's algorism, $f_{r-1}$ is a factor of $f_{r-2}$, by the preceding equation it is a factor of $f_{r-3}$, . . . , and hence it is a factor also of $f_1$ and $f$. Moreover, it is the greatest common factor of $f$ and $f_1$ since it contains every common factor of $f$ and $f_1$, and so it follows that

(2) **the greatest common factor of two integral functions is computable by rational operations.**

Substituting the value of $f_2$ obtained from the first equation of Euclid's algorism into the second, we find

$$f_3 = (1 + q_1 q_2) f_1 - q_2 f,$$

or abbreviated:

$$f_3 = g_1 f + g f_1.$$

Expressing by similar substitutions the value of every following $f_i$ in terms of $f$ and $f_1$, we finally obtain

$$f_r = G_1 f + G f_1.$$

In this equation $G$ and $G_1$ are integral functions of $x$, but they are not uniquely defined since the equation remains true when we replace $G$ by $G + Hf$ and $G_1$ by $G_1 - Hf_1$, where $H$ is any integral function of $x$. There is only one function $G_1$, however, of degree not more than $n_1 - 1$, so that $f_r - G_1 f$ is of degree not more than $n + n_1 - 1$, if $n$ and $n_1$ are the degrees of $f$ and $f_1$; and corresponding to this only one function $G$ of degree not more than $n - 1$.

If $f$ and $f_1$ have no common factor, $f_r$ is a constant different from zero, and by dividing it out we prove:

(3) **If two integral functions $f$ and $f_1$ of degree $n$ and $n_1$ have no common factor, we can compute by rational operations two integral functions $\varphi$ and $\varphi_1$ which satisfy identically the relation**

$$\boxed{\varphi_1 . f + \varphi . f_1 = 1}$$

**and are uniquely defined of degree not more than $n - 1$ and $n_1 - 1$, if we so choose.**

Multiplying this equation by any integral function $g$ of $x$, we have

$$g\varphi_1 f + g\varphi f_1 = g,$$

and setting

$$g\varphi = qf + r,$$

where $q$ and $r$ are integral functions of $x$ and $r$ is of degree smaller than $n$, we have

$$g\varphi_1 f + qff_1 + rf_1 = g,$$

where we may put:

$$r_1 = g\varphi_1 + qf_1.$$

From this we conclude:

(4) **If $f$ and $f_1$ and $g$ are integral functions of $x$ such that $f$ and $f_1$ have no common factor, we can compute by rational operations two integral functions $r$ and $r_1$ of $x$ which satisfy identically the relation**

$$\boxed{r_1 . f + r . f_1 = g},$$

**and $r$ such that it is of lower degree than $f$.**

## §3. REDUCTION

An integral function of $x$ is said to be **reducible** if it is expressible as product of some integral functions of $x$ other than constants and **irreducible** if it is not expressible so.

The following proposition is quickly verified:

(5) **If an integral function of $x$ divides the product of two other integral functions of $x$, then, having no factor in common with one of them, it divides the other.**

For if the integral function $f$ of $x$ divides the product of two integral functions $g$ and $h$ of $x$ and has no factor in common with $g$, we can find by proposition (3) two integral functions $\varphi$ and $\psi$ of $x$ such that

$$\varphi f + \psi g = 1,$$

and from

$$\varphi fh + \psi gh = h$$

follows that $f$ divides $h$ since it divides both terms on the left.

The proposition is readily extended to include the product of more than two functions and as such leads to another:

(6) **A reducible function of $x$ is expressible as product of irreducible factors in only one way, if we disregard constants.**

For suppose the reducible function $F$ of $x$ is expressible so in two ways:

$$F = f_1f_2 \ldots = g_1g_2 \ldots$$

Then the irreducible function $g_1$ must divide some function $f_i$, say $f_1$, and differ from it by a constant factor alone since $f_1$ also is irreducible:

$$f_1 = k_1g_1.$$

From

$$k_1f_2 \ldots = g_2 \ldots$$

follows likewise that $g_2$ differs from $f_2$, say, by a constant factor alone:

$$f_2 = k_2g_2,$$

and so on. This proves the proposition.

## §4. PRIMITIVE FUNCTION

If an integral function has integral coefficients, the greatest common factor of those coefficients is called the **divisor of the function,** and if the divisor is 1, the function is said to be **primitive.**

Let

$$g = g_0x^m + g_1x^{m-1} + \ldots + g_m$$

and

$$h = h_0x^n + h_1x^{n-1} + \ldots + h_n$$

be two primitive functions and

$$F = a_0x^{m+n} + a_1x^{m+n-1} + \ldots + a_{m+n}$$

their product. Suppose that the coefficients

$$g_0, g_1, \ldots, g_{k-1}$$
$$h_0, h_1, \ldots, h_{l-1}$$

are divisible by some prime number $p$, while the coefficients $g_k$ and $h_l$ are not. Then

$$a_{k+l} = g_kh_l + g_{k-1}h_{l+1} + \ldots$$
$$+ g_{k+1}h_{l-1} + \ldots$$

is not divisible by $p$ either, inasmuch as all its terms except the first one are divisible. Hence $F$ is primitive and it appears that

(7) **the product of primitive functions is primitive.**

Since the product of the imprimitive functions $pg$ and $qh$, where $p$ and $q$ are integers, evidently is divisible by $pq$, we may add that

(8) **the divisor of a product of imprimitive functions is the product of the divisors of those functions.**

If an integral function of the form

$$x^n + a_1x^{n-1} + \ldots + a_n$$

has fractional coefficients, we can represent it as fraction of a primitive function bringing its terms over the lowest common denominator. It follows by proposition (7) that the product of two such functions is of the same form and hence also has fractional coefficients.

To recognize a primitive function

$$f = x^n + a_1x^{n-1} + \ldots + a_n,$$

where $x^n$ has the coefficient

$$a_0 = 1,$$

as irreducible, the following proposition[1] is helpful:

**(9) If the coefficients $a_i$ other than $a_0$ of a primitive function $f$ are divisible by a prime number $p$ while $a_0 = 1$ and $a_n$ is not divisible by $p^2$, the function $f$ is irreducible.**

For suppose that

$$f = gh,$$

where

$$\begin{aligned} g &= x^k + g_1x^{k-1} + \cdots + g_k \\ h &= x^l + h_1x^{l-1} + \cdots + h_l. \end{aligned}$$

Then $g$ and $h$ have integral coefficients since $f$ has.

Comparing coefficients, we set

$$\begin{aligned} a_n &= g_kh_l \\ a_{n-1} &= g_kh_{l-1} + g_{k-1}h_l \\ a_{n-2} &= g_kh_{l-2} + g_{k-1}h_{l-1} + g_{k-2}h_l \\ &\cdots\cdots\cdots \\ a_{n-k} &= g_kh_{l-k} + g_{k-1}h_{l-k+1} + \cdots + g_1h_{l-1} + h_l, \end{aligned}$$

where the $h_i$ with possible negative subscripts are 0 and where

$$h_0 = 1.$$

If now every $a_i$ other than $a_0$ is divisible by $p$ and $a_n$ not by $p^2$, we infer from the first equation that either $g_k$ or $h_l$ is divisible by $p$ and the other is not. But when $g_k$ is divisible by $p$, we infer from the following equations that $g_{k-1}, g_{k-2}, \ldots, g_1$ are divisible by $p$, and from the last equation that $h_l$ is divisible so, which is a contradiction.

As the same argument applies when $h_l$ is divisible by $p$, the proposition is proved.

## §5. LINEAR FACTORS

Dividing an integral function

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

by the linear function $x - \alpha$, we obtain

$$f(x) = (x - \alpha)g(x) + c,$$

[1] Called the proposition of Eisenstein.

where the remainder $c$ is a constant since as function of $x$ it is of lower degree than $x - \alpha$. Substituting from

$$g(x) = b_0x^{n-1} + b_1x^{n-2} + \dots + b_{n-2}x + b_{n-1}$$

with degree one less than the degree of $f(x)$, we have

$$\begin{aligned} f(x) = b_0x^n &+ b_1x^{n-1} + \dots + b_{n-1}x + c \\ &- \alpha b_0x^{n-1} - \dots - \alpha b_{n-2}x - \alpha b_{n-1}, \end{aligned}$$

whence

$$\begin{aligned} b_0 \quad\quad &= a_0 \\ b_1 - \alpha b_0 &= a_1 \\ \dots\dots &\dots \\ b_{n-1} - \alpha b_{n-2} &= a_{n-1} \\ c - \alpha b_{n-1} &= a_n \end{aligned}$$

and consequently

$$\begin{aligned} b_0 &= a_0 \\ b_1 &= a_0\alpha + a_1 \\ \dots &\dots \\ b_{n-1} &= a_0\alpha^{n-1} + a_1\alpha^{n-2} + \dots + a_{n-1} \\ c &= a_0\alpha^n + a_1\alpha^{n-1} + \dots + a_{n-1}\alpha + a_n. \end{aligned}$$

Comparing now $f(x)$ with $c$, we find that

$$c = f(\alpha);$$

hence

$$g(x) = \frac{f(x) - f(\alpha)}{x - \alpha},$$

which for $x = \alpha$ gives

$$g(\alpha) = f'(\alpha)$$

if we assume here the continuity of integral functions[1] and introduce the derivative $f'(x)$ of $f(x)$.

This clears the way for the discussion of roots. When

$$f(\alpha) = 0,$$

we call $\alpha$ a **root** of the function $f(x)$ and have from the preceding

$$f(x) = (x - \alpha)g(x).$$

[1] The continuity of integral functions underlies also the fundamental theorem of algebra which is used on the next page. For the continuity of integral functions cf. Townsend, Functions of a Complex Variable, Ch. II, and Burnside-Panton, Theory of Equations, Art. 7, p. 9, and Art. 192, p. 427.

By the **fundamental theorem of algebra,**[1] every integral function has a root, whence it follows as before that

$$g(x) = (x - \beta)g_1(x)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$g_{n-2}(x) = a_0(x - \nu);$$

consequently

$$f(x) = a_0(x - \alpha)(x - \beta) \ . \ . \ . \ (x - \nu).$$

It appears that

(10) **an integral function of degree $n$ is the product of $n$ linear factors,**

and that

(11) **an integral function of degree $n$ has just $n$ roots.**

It could not have more roots unless it would vanish identically with coefficients equal to zero, for the degrees of $g, g_1, \ . \ . \ .$ constantly decrease by one.

Since we have

$$g(\alpha) = a_0(\alpha - \beta) \ . \ . \ . \ (\alpha - \nu) = f'(\alpha),$$

it follows that

(12) **the function $f(x)$ has a multiple root $\alpha$ when**

$$\boxed{f(\alpha) = f'(\alpha) = 0},$$

that is to say when not only the function but also its derivative vanishes for $\alpha$.

[1] See footnote on preceding page. For the fundamental theorem of algebra cf. Townsend, Functions of a Complex Variable, Art. 54, p. 291, and Burnside-Panton, Theory of Equations, Art. 195, p. 431; see also the end of §82.

# CHAPTER II

## EQUATIONS AND PERMUTATIONS

### §6. DISCOVERY OF LAGRANGE

Easy as the solution of the linear equation is, and versed as we are in the solution of the quadratic, the solution of equations of higher degrees becomes increasingly difficult if not impossible. And it is the study of these equations which gave to us the celebrated theories of Lagrange and Galois and with them the first investigations into **groups** and the first dim notion of **domains**: two concepts that were destined to predominate in modern algebra.

We consider a discovery of Lagrange. Given the general cubic equation

$$x^3 - bx^2 + cx - d = 0$$

whose roots we denote by $x_\iota$, we set

$$x = y + \frac{b}{3}$$

and obtain the reduced cubic equation

$$y^3 + py - q = 0,$$

where

$$p = c - \frac{b^2}{3}$$

and

$$q = d - \frac{bc}{3} + \frac{2b^3}{27}.$$

Its roots are by **Cardan's formula**

$$y_1 = \sqrt[3]{\frac{q}{2} + \sqrt{R}} + \sqrt[3]{\frac{q}{2} - \sqrt{R}}$$

$$y_2 = \omega^2 \sqrt[3]{\frac{q}{2} + \sqrt{R}} + \omega \sqrt[3]{\frac{q}{2} - \sqrt{R}}$$

$$y_3 = \omega \sqrt[3]{\frac{q}{2} + \sqrt{R}} + \omega^2 \sqrt[3]{\frac{q}{2} - \sqrt{R},}$$

where

$$R = \frac{q^2}{4} + \frac{p^3}{27}$$

and where

$$\omega = \frac{-1 + \sqrt{-3}}{2}, \quad \omega^2 = \frac{-1 - \sqrt{-3}}{2}$$

are primitive cube roots[1] of unity for which

$$\omega^2 + \omega + 1 = 0,$$
$$\omega^3 = 1.$$

To find $x_i$, we only have to add $b/3$ to $y_i$; whence in the expressions for $x_i$ there will be no radicals save those contained in the expressions for $y_i$. And these radicals Lagrange noticed to be expressible as functions of the roots $x_i$ themselves.

To see this for the cube roots of the formula, we multiply the equations for the $y_i$ by 1, $\omega$, $\omega^2$ respectively and adding them obtain

$$3\sqrt[3]{\frac{q}{2} + \sqrt{R}} = y_1 + \omega y_2 + \omega^2 y_3.$$

Substituting then for $y_i = x_i - \frac{b}{3}$, we have

$$3\sqrt[3]{\frac{q}{2} + \sqrt{R}} = x_1 + \omega x_2 + \omega^2 x_3.$$

This equation has a cube root as member. Setting

$$\varphi_1 = x_1 + \omega x_2 + \omega^2 x_3 = 3\sqrt[3]{\frac{q}{2} + \sqrt{R}},$$

we know from elementary algebra that the other two cube roots are

$$\varphi_3 = \omega^2 \varphi_1 = 3\omega^2 \sqrt[3]{\frac{q}{2} + \sqrt{R}}$$

and

$$\varphi_5 = \omega \varphi_1 = 3\omega \sqrt[3]{\frac{q}{2} + \sqrt{R}}.$$

Again, multiplying the equations for the $y_i$ by 1, $\omega^2$, $\omega$ respectively, adding them and substituting for $y_i$, we obtain

$$3\sqrt[3]{\frac{q}{2} - \sqrt{R}} = x_1 + \omega^2 x_2 + \omega x_3.$$

[1] Primitive roots are defined in §§38, 79 and computed in §84.

And if we set

$$\varphi_2 = x_1 + \omega^2 x_2 + \omega x_3 = 3\sqrt[3]{\frac{q}{2} - \sqrt{R}},$$

the other two cube roots are

$$\varphi_4 = \omega\varphi_2 = 3\omega\sqrt[3]{\frac{q}{2} - \sqrt{R}},$$

and

$$\varphi_6 = \omega^2\varphi_2 = 3\omega^2\sqrt[3]{\frac{q}{2} - \sqrt{R}}.$$

To find $\sqrt{R}$ in terms of the roots $x_i$, we cube the equations for $\varphi_1$ and $\varphi_2$ and subtract. We obtain

$$54\sqrt{R} = (x_1 + \omega x_2 + \omega^2 x_3)^3 - (x_1 + \omega^2 x_2 + \omega x_3)^3,$$

and after expanding, simplifying and factoring we have

$$18\sqrt{R} = \sqrt{-3}\,(x_1 - x_2)(x_1 - x_3)(x_2 - x_3),$$

where

$$\sqrt{-3} = \omega - \omega^2.$$

The other square root is $-\sqrt{R}$.

## §7. SOLUTION OF CUBIC

The solution of the general cubic equation requires the computation of the six values $\varphi_i$, but is then readily found from the equations

$$\begin{aligned}\varphi_1 &= x_1 + \omega x_2 + \omega^2 x_3\\ \varphi_2 &= x_1 + \omega^2 x_2 + \omega x_3\\ b &= x_1 + x_2 + x_3\end{aligned}$$

if we add them as they stand, and multiplied by $\omega^2$, $\omega$, 1, and then by $\omega$, $\omega^2$, 1 respectively. It may be found also from two other values $\varphi_i$ if we take care to choose them so that their product is equal to

$$3\sqrt[3]{\frac{q}{2} + \sqrt{R}} \cdot 3\sqrt[3]{\frac{q}{2} - \sqrt{R}} = -3p,$$

which is the product of $\varphi_1$ and $\varphi_2$.

We can compute the six values $\varphi_i$ when we know the two values of their cubes. Setting

$$A_1 = \frac{q}{2} + \sqrt{R}$$

and

$$A_2 = \frac{q}{2} - \sqrt{R},$$

we have the equations

$$\left(\frac{\varphi_j}{3}\right)^3 = A_1 \qquad [j = 1, 3, 5$$

and

$$\left(\frac{\varphi_k}{3}\right)^3 = A_2 \qquad [k = 2, 4, 6$$

to find the $\varphi_i$. The $A_i$ we then recognize as the roots of the quadratic equation

$$A^2 - qA - \frac{p^3}{27} = 0.$$

This last equation cannot be avoided either for the six values $\varphi_i$ satisfy the equation

$$(\varphi - \varphi_1)(\varphi - \varphi_2)(\varphi - \varphi_3)(\varphi - \varphi_4)(\varphi - \varphi_5)(\varphi - \varphi_6) = 0,$$

and multiplying out we obtain the quadratic equation

$$\varphi^6 - (\varphi_1^3 + \varphi_2^3)\varphi^3 + \varphi_1^3\varphi_2^3 = 0$$

in $\varphi^3$ which is reducible[1] to the other in $A$.

The coefficients $q$ and $p^3/27$ which permit us to compute the two values $A_i$ and the six values $\varphi_i$ have just one value as they are rational functions of the coefficients $d$, $c$, $d$.

Thus, retracing the solution of the general cubic equation, we begin with its coefficients of one definite value. We solve a rational quadratic equation and find the two values $A_i = q/2 \pm \sqrt{R}$, introducing or **adjoining** the radical $\sqrt{R}$ not contained in the coefficients. Then we find the six values $\varphi_i$ extracting the cube roots of $A_i$, and adjoining them we compute the roots $x_i$ of the cubic.

[1] Cf. Serret's Algebra No. 508.

## §8. CONNECTION WITH PERMUTATIONS

All the quantities that we operate with in solving the general cubic equation are expressible as functions of the roots $x_i$ themselves of the cubic.

Tabulating all $\varphi_i$ as functions of the $x_i$ and rearranging the order of their terms, we have

$$\begin{aligned}
\varphi_1 &= x_1 + \omega x_2 + \omega^2 x_3 \\
\varphi_3 &= x_2 + \omega x_3 + \omega^2 x_1 = \omega^2 \varphi_1 \\
\varphi_5 &= x_3 + \omega x_1 + \omega^2 x_2 = \omega \varphi_1 \\
\varphi_2 &= x_1 + \omega x_3 + \omega^2 x_2 \\
\varphi_4 &= x_2 + \omega x_1 + \omega^2 x_3 = \omega \varphi_2 \\
\varphi_6 &= x_3 + \omega x_2 + \omega^2 x_1 = \omega^2 \varphi_2.
\end{aligned}$$

If we now closely examine these functions, we perceive that all values $\varphi_i$ are obtained from $\varphi_1$ by interchanging in all possible ways the letters $x_i$. Every such permutation changes the function $\varphi_1$, and corresponding to the $3! = 6$ possible permutations we have six values $\varphi_i$. Hence $\varphi_1$ is said to be a six-valued function of the $x_i$.

The radical $\sqrt{R}$ as function of the $x_i$ is by §6 a multiple of

$$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

Interchanging here the letters $x_i$ in all possible ways, we obtain just two alternating values with the plus and minus sign, hence $\sqrt{R}$ is said to be a two-valued or **alternating function**[1] of the $x_i$.

The coefficients of the general cubic equation are as functions of the $x_i$:

$$\begin{gathered}
x_1 + x_2 + x_3 \\
x_1x_2 + x_1x_3 + x_2x_3 \\
x_1x_2x_3.
\end{gathered}$$

Interchanging the letters $x_i$ in all possible ways, we do not alter the value of the coefficients, hence they are said to be one-valued or **symmetric functions**[2] of the $x_i$.

The function $A_i$, composed of a one-valued and a two-valued function, is obviously two-valued.

Thus we notice the existence of functions in the roots $x_i$ of the cubic equation which are one-valued, two-valued and six-

[1] Cf. §29.
[2] Cf. §22.

valued with regard to the permutations on these roots. We notice the possibility of computing two-valued functions from one-valued functions by adjoining square roots, and of six-valued functions from two-valued functions by adjoining cube roots. We notice the possibility of expressing the roots $x_i$ themselves rationally in terms of such functions with the highest number of values.

Does all this not suggest a connection between the solution of an equation and the permutations on its roots? It did suggest that to Lagrange.

# CHAPTER III

## ALGEBRA OF PERMUTATIONS

### §9. NOTATION

To interchange symbols is to perform on them a **permutation or substitution.** It is commonly denoted by $s$ or a letter following $s$.

Cauchy calls such an operation a substitution and the result of it a permutation. But the use of these terms is not settled as yet, and it seems preferable to denote by substitution a different operation, as it occurs in §63.

We shall operate permutations on letters $x_i$ representing roots of an equation; and, more seldom, on functions of these $x$, denoting such a permutation by $\sigma$.

The $n$ letters $x_i$ of the general case are roots of the general equation

$$f(x) = a_0x^n + a_1x^{n-1} + \ . \ . \ . + a_n = 0.$$

Given the three letters

$$x_1, x_2, x_3,$$

we can perform on them altogether 3! permutations, including the permutation that leaves the arrangement of the letters unaltered. This last one we call the **identical permutation or identity** and denote by 1, if no confusion can arise.

To agree upon a notation for permutations other than identity, we suppose that some permutation $s$ changes the arrangement

$$x_1, x_2, x_3$$

into the arrangement

$$x_2, x_3, x_1.$$

We could denote this permutation by

$$s = \begin{pmatrix} x_1x_2x_3 \\ x_2x_3x_1 \end{pmatrix},$$

meaning that each letter in the upper row is replaced by the corresponding letter in the lower row. Or, simpler, could denote it by

$$s = \begin{pmatrix} 123 \\ 231 \end{pmatrix},$$

writing only the subscripts of the $x_i$.

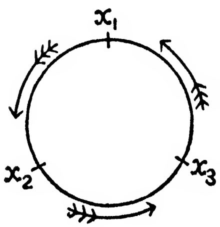
But we notice that the permutation $s$ replaces $x_1$ by $x_2$, $x_2$ by $x_3$, $x_3$ by $x_1$; that is, replaces the letters in a cycle. And we readily find a more convenient and frequent notation:

$$s = (x_1x_2x_3),$$

or

$$s = (123),$$

meaning that each letter in the **cycle** is replaced by the following one and the last letter by the first one.




Such a permutation, interchanging the letters $x_i$ cyclically, is called a **circular permutation.** It can suitably be illustrated by a circular diagram, and it should be clear that in the **circular notation** the symbols

$$(123),\ (231),\ (312)$$

represent the same permutation, which can be denoted in as many ways as there are letters marked in its cycle.

But not a circular permutation only, any permutation can be represented in the circular notation if we decompose it into cycles. We do that taking any one letter and noting in a cycle the succession of replacing letters until we return to the letter that we started from. Here we close the cycle and begin a new one with a letter not yet noted, and we keep on doing so until the notation is complete. Thus we may have a permutation $t$ on five letters $x_i$ denoted as

$$t = \begin{pmatrix} x_1x_2x_3x_4x_5 \\ x_3x_4x_5x_2x_1 \end{pmatrix} = (x_1x_3x_5)(x_2x_4) = (135)(24).$$

## §10. DEGREE

The number of letters operated on by a permutation is its **degree.** The permutation

$$s = (123)$$

is of degree three, the permutation

$$t = (135)(24)$$

is of degree five. The **degree of a cycle** is the number of letters marked in a cycle. Thus we have to distinguish between the degree of a permutation and the degrees of its cycles: the permutation $t$, while of degree five, has cycles of degree three and two.

A cycle of degree one notes that the letter marked in the cycle has not been displaced. It is commonly omitted in the notation and not counted toward the degree: we set for instance

$$(135)(2)(4) = (135).$$

A permutation of degree two, as

$$(12),$$

is called a **transposition.** A permutation whose cycles are all of the same degree, as

$$(123)(456),$$

is called **regular.** Two permutations with cycles corresponding as to number and degree, as

$$(123)(45) \text{ and } (135)(24),$$

are called **similar.**

## §11. COMBINATION

Permutations obey the **law of combination**: when we have two permutations $s$ and $t$ and apply them successively, we combine or **multiply** them, as we say, into a permutation denoted by $st$ and called the **product** of $s$ and $t$, and there is only one such product.

For example, to the set of letters

$$x_1, x_2, x_3$$

we apply successively the permutations

$$s = (123)$$

and

$$t = (23).$$

The given arrangement of the $x_i$ is changed by $s$ into

$$x_2, x_3, x_1,$$

and this by $t$ into

$$x_3,\ x_2,\ x_1.$$

This result we obtain directly by applying to the given arrangement of the $x_i$ the permutation

$$\begin{pmatrix}123\\321\end{pmatrix} = (13);$$

hence

$$st = (123)(23) = (13).$$

This product of $s$ and $t$ can be read off from the permutations $s$ and $t$ without performing the indicated operations: $s$ replaces 1 by 2 and $t$ replaces 2 by 3, therefore $st$ replaces 1 by 3; $s$ replaces 3 by 1 and $t$ does not alter 1, therefore $st$ replaces 3 by 1 and 3 closes the cycle; $s$ replaces 2 by 3 and $t$ replaces 3 by 2, hence 2 is not displaced by $st$.

Similarly, in four letters $x_i$ we find that

$$(12)(13)(14) = (1234).$$

Applying first the permutation $t$ and then the permutation $s$, we obtain the product $ts$ of those permutations which is not the same as the product $st$:

$$ts = (23)(123) = (12).$$

Hence it appears that permutations do not in general obey the **commutative law.** Yet they may do so, in which case they are called **commutative or permutable**: for instance, the permutations

$$s = (12)(34)$$

and

$$t = (13)(24)$$

on four letters $x_i$ give

$$st = ts = (14)(23).$$

It is obvious that permutations are always commutative when they operate on different letters: for instance, the permutations

$$s = (123)$$

and

$$t = (45)$$

on five letters $x_i$ give

$$st = ts = (123)(45) = (45)(123).$$

It follows that

(13) **a non-circular permutation can always be represented as product of commutative circular permutations.**

## §12. ORDER

Multiplying the permutation $s$ by itself, we obtain the products $ss$, $sss$, . . . , and adopting a usage of elementary algebra, we call them **powers** of $s$ and set

$$ss = s^2, \quad s^2s = s^3, \ldots$$

If

$$s = (123),$$

then $s^2$ replaces 1 by 2 and 2 by 3, 3 by 1 and 1 by 2, 2 by 3 and 3 by 1, hence

$$s^2 = (132).$$

Likewise, $s^3 = s^2s$ replaces 1 by 3 and 3 by 1, 2 by 1 and 1 by 2, 3 by 2 and 2 by 3; hence

$$s^3 = (1)(2)(3) = 1.$$

Continuing the involution on $s$, we make up the table

$$\begin{aligned} 1 &= s^3 = s^6 = \ldots \\ s &= s^4 = s^7 = \ldots \\ s^2 &= s^5 = s^8 = \ldots \end{aligned}$$

and find that we can obtain no other distinct permutations than 1, $s$, $s^2$ and that the permutations $s^3$, $s^6$, . . . are equal to identity.

The involution on

$$t = (1234)$$

gives

$$\begin{aligned} t^2 &= (13)(24) \\ t^3 &= (1432) \\ t^4 &= (1)(2)(3)(4) = 1. \end{aligned}$$

We notice that the powers of a circular permutation are not all circular unless the degree of the permutation is prime, but they are all regular.

The lowest power of a permutation or a cycle that equals identity, or the difference of two successive powers that are equal, is the **order or period** of the permutation or the cycle.

Thus the permutation $s = (123)$ is of order three, the permutation $t = (1234)$ is of order four.

To form the square of a circular permutation we replace each letter by the second to its right, to form the cube, by the third. If we have a circular permutation $s$ of degree $n$, the $n$-th power of $s$, replacing each letter by the $n$-th to its right, replaces each letter by itself and is identity.

Any power of such a permutation $s$ higher than $n$ equals some power of $s$ lower than $n$, and all distinct permutations that we can obtain by involution on $s$ are contained in the set

$$1,\ s,\ s^2,\ \ldots,\ s^{n-1},$$

or

$$s,\ s^2,\ s^3,\ \ldots,\ s^n = 1.$$

Hence it appears that

(14) **the order of a circular permutation is equal to its degree, and both are determined by the number of letters acted upon by the permutation.**

To find the order of a non-circular permutation, we decompose it into cycles; to the cycles we then apply the proposition for circular permutations. For instance, the permutation

$$u = (123)(45)$$

gives

$$\begin{aligned} u^2 &= (132)(4)(5), & u^3 &= (1)(2)(3)(45), \\ u^4 &= (123)(4)(5), & u^5 &= (132)(45), \\ u^6 &= (1)(2)(3)(4)(5) = 1 & & \end{aligned}$$

and is of order six.

Clearly then,

(15) **the order of a non-circular permutation is the lowest common multiple of the orders of its cycles,**

for only such a power of a non-circular permutation resolved into cycles turns each cycle into identity.

It follows that the order of a permutation cannot be prime unless the permutation is circular of prime degree or regular with cycles of prime degree. Such permutations are, for instance:

$$\begin{aligned} s &= (123) & v &= (123)(456) \\ s^2 &= (132) & v^2 &= (132)(465) \\ s^3 &= 1 & v^3 &= 1. \end{aligned}$$

Hence we can note:

(16) **If the order of a permutation is prime, its powers other than identity are similar permutations,**

which is readily verified. Furthermore:

(17) **If order and degree of a permutation are prime, the permutation is circular,**

for a regular permutation with several cycles cannot be of prime degree.

## §13. ASSOCIATION

While the commutative law does not generally hold for permutations, the **associative law** does:

$$(st)u = s(tu),$$

and we write for both $stu$. Likewise, we have

$$(st)(uv) = (stu)v = s(tuv) = s(tu)v = stuv.$$

We can test the associative law on any example, but it also follows for permutations from the law of combination. This law, requiring that the product of two permutations be identically defined, can be extended to the product of three permutations. But the simplest way of expressing the postulate that such a product be identically defined is the associative law.

From the associative law follows that

$$s^k s^l = s^l s^k = s^{k+l},$$

as we see from the scheme

$$\underbrace{\overbrace{ss \ldots s\, s \ldots s}^{k}}_{}\!\!\overbrace{s \ldots s.}^{l}$$

$$\underbrace{ss \ldots s}_{l}\,\underbrace{s \ldots s\, s \ldots s.}_{k}$$

Hence powers of a permutation are commutative.

If the two permutations $s$ and $t$ are commutative, we can set

$$s^k t^l = t^l s^k.$$

This we show to be true for the particular case $s^3t^2$, by induction it will then be true in general:

$$s^3t^2 = ssstt = sstst = ststs = tstss = ttsss = t^2s^3.$$

We can also set

$$(st)^2 = s^2t^2$$

if the permutations $s$ and $t$ are commutative; but not otherwise, since

$$(st)^2 = stst = sstt = s^2t^2.$$

To set

$$(stu\ .\ .\ .\ )^2 = s^2t^2u^2\ .\ .\ .\ ,$$

it is necessary that every pair of permutations be commutative.

## §14. INVERSE

There always exists a permutation that undoes or reverses the interchange of letters effected by another permutation, and it is called the **inverse** of that permutation. The product of a permutation and its inverse evidently is identity. Again, if the product of two permutations is identity, each permutation is inverse to the other. For instance, the permutations

$$s = \binom{123}{231} = (123)$$

and

$$t = \binom{231}{123} = (132)$$

are each inverse to the other, and

$$st = 1.$$

It is clear that we obtain the inverse of a permutation by reversing the order of the numbers in its cycles. The example makes this evident for circular permutations:

$$s = (123)$$
$$t = (321),$$

and for non-circular permutations it follows by proposition (13). We infer that

(18) **the inverse of a permutation $s$ is a permutation similar to $s$.**

If the order of a permutation $s$ is $r$, the permutations $s^k$ and $s^{r-k}$ are each inverse to the other because

$$s^ks^{r-k} = 1.$$

If the order is three, $s^2$ and $s$ are each inverse to the other; if the order is two, $s$ is its own inverse.

Adopting a usage of elementary algebra, we denote the inverse of a permutation $s$ by $s^{-1}$, so that

$$ss^{-1} = 1$$

and

$$t = s^{-1}$$

if

$$st = 1.$$

We adopt also the elementary notation

$$s^{-1}s^{-1} = s^{-2}, \quad s^{-2}s^{-1} = s^{-3}, \ . \ . \ .$$

and the symbols $s^1$ and $s^0$ defined by the relations

$$s^1 = s$$

and

$$s^0 = 1,$$

so that

$$ss^{-1} = s^1s^{-1} = s^0 = 1.$$

The inverse of $s^2$ is $s^{-2}$ because

$$s^2s^{-2} = s(ss^{-1})s^{-1} = 1,$$

and a like relation is true for any power.

The inverse of $st$, denoted by $(st)^{-1}$, is $t^{-1}s^{-1}$ and not $s^{-1}t^{-1}$, for only

$$st \, . \, t^{-1}s^{-1} = 1;$$

having operated the permutation $st$ we must retrace our way back step by step.

With the indices, we define also **division** by a permutation as elementary algebra would have it, namely as multiplication by the inverse of the permutation. Thus

$$\frac{s^3}{s^2} = s^3 \, . \, s^{-2} = s.$$

If

$$s^3 = 1,$$

multiplication by $s$ is equivalent to division by $s^2$; which we readily interpret on the circular diagram of §9, taking division and the negative sign of an exponent as reversing direction.

## §15. IDENTITY

The multiplication of a permutation on either side by identity obviously does not give a product different from that permutation:

$$1 \,.\, s = s \,.\, 1 = s.$$

Identity may therefore be regarded as commutative with every permutation and may be suppressed in the notation of a product.

Conversely,

(19) **if $e$ and $s$ are two permutations such that**

$$\boxed{es = se = s},$$

**then $e$ is identity.**

For

$$1 \,.\, e = e$$

from the nature of identity, and

$$1 \,.\, e = 1$$

since it is given that $se = s$. Hence it follows that

$$e = 1.$$

We conclude with a few examples:

(a) Prove that $t = u$ if $st = su$:

$$s^{-1} \,.\, (st) = s^{-1} \,.\, (su); \quad (s^{-1}s)t = (s^{-1}s)u; \quad t = u.$$

This verifies that $e = 1$ if $es = se = s$, for $s1 = s$ and $se = s$ give $se = s1$ and consequently $e = 1$.

(b) Prove that $st$ is the inverse of $ts$ if $s^2 = t^2 = 1$:

$$st \,.\, ts = st^2s = s^2 = 1$$

It follows that

$$st = (ts)^{-1} = s^{-1}t^{-1}.$$

(c) Prove that $s$ and $t$ are commutative if $s^2 = t^2 = (st)^2 = 1$:

$$st = t^2st = ttst = ts^2tst = tsstst = ts(st)^2 = ts.$$

(d) Prove that $h_2 = s^{-1}h_1s$ if $sh_2s^{-1} = h_1$:

$$sh_2s^{-1} = h_1; \quad s^{-1}sh_2s^{-1} = s^{-1}h_1; \quad h_2s^{-1}s = s^{-1}h_1s; \quad h_2 = s^{-1}h_1s.$$

## CHAPTER IV

## GROUP AND SUBGROUP

### §16. GROUP

Suppose we have $n$ letters $x_i$. By functions of such letters we always mean rational functions, both here and later.

If to a function $\varphi$ of the $x_i$ we apply all permutations that are possible between the $x_i$, there will be some that do not alter the function $\varphi$. Let them compose the set

$$s_1, s_2, \ . \ . \ . \ , s_r.$$

A complete set of distinct permutations that do not alter a function is called a **group of permutations** and is denoted by $G$ unless we use special notation. It will often be found convenient to mark $G$ as a group by noting $\{G\}$. The development of the group theory is due to the genius of Cauchy.[1]

A function $\varphi$ is said to belong to a group of permutations if it remains unaltered by all those, and only those, permutations which are in the group. It appears that

(20) **every function $\varphi$ of the $x_i$ belongs to a group $G$ of permutations between the $x_i$.**

Since identity is contained in every group of permutations, we commonly set

$$G = 1, s_2, \ . \ . \ . \ , s_r,$$

assuming that

$$s_1 = 1.$$

When we apply two permutations of $G$ successively to the function $\varphi$, or one permutation twice, the operation leaves $\varphi$ unaltered since every permutation of $G$ does so.

It follows that a group of permutations contains the product of any permutation by itself or another permutation, hence in

[1] Cauchy lived 1789–1857.

particular the inverse of any permutation. And this property defines a group if we can prove that

(21) **to every group $G$ of permutations between the $x_i$ belongs a function $\varphi$ of the $x_i$.**

The function

$$\psi_1 = \alpha_1 x_1 + \alpha_2 x_2 + \ . \ . \ . \ + \alpha_n x_n$$

is evidently altered by every permutation of $G$ other than identity, and by no two permutations alike. Suppose that the permutations

$$1, s_2, \ . \ . \ . \ , s_r$$

of $G$ convert $\psi_1$ into

$$\psi_1, \psi_2, \ . \ . \ . \ , \psi_r$$

respectively. We form the function

$$\varphi = \psi_1 + \psi_2 + \ . \ . \ . \ + \psi_r;$$

this function belongs to $G$ and satisfies the proposition.

For any permutation $s_i$ of $G$ applied to $\varphi$ gives

$$\varphi_i = \psi_{1i} + \psi_{2i} + \ . \ . \ . \ + \psi_{ri},$$

where the subscripts identify the permutations that have been applied to the functions. But each of the permutations

$$1s_i, s_2 s_i, \ . \ . \ . \ , s_r s_i$$

is in $G$ since it is the product of two permutations which are in $G$, and no two of these permutations are alike since from

$$s_j s_i = s_k s_i$$

would follow

$$s_j = s_k,$$

which is untrue. These permutations represent therefore in some order or other the permutations of $G$, and hence the $\psi_{ki}$ in some order or other the $\psi_i$. Consequently

$$\varphi_i = \varphi.$$

If a permutation $t$ is not in $G$, then $s_i\, t$ is not in $G$ either, and

$$\varphi_t \neq \varphi.$$

The function $\varphi$, remaining unaltered by every permutation of $G$ and such a permutation alone, belongs to $G$. And since we

can construct in the way indicated any number of functions belonging to $G$,

(22) **a group of permutations is defined as a set of distinct permutations such that the product of any two, or the square of any one, is a permutation of the set.**

The number of permutations contained in a group is its **order**, the number of letters that its permutations operate on is its **degree.**

The degree of a group $G$ is denoted by $n$ which may be added to the symbol of the group as in $G^n$.

The order of a group $G$ is denoted by $r$, or by $r_g$ to identify the group it belongs to, and may be added to the symbol of the group as in $G_r$.

The simplest group in a sense is **identity** by itself. It is denoted by 1 if no confusion can result, otherwise by $\{1\}$, and its order is one. In $\psi_1$ we had a function belonging to identity.

Next in simplicity is a group containing only the powers of a permutation $s$. It is called a **cyclic group** and denoted by $C$ or

$$\{s\} = s, s^2, \ldots s^r;$$

its order is the order of the permutation $s$.

The complete group of all $n!$ permutations that are possible between $n$ letters $x_i$ is called the **symmetric group** on these letters. It is the group of symmetric functions,[1] is denoted by $S$, and $n!$ is its order.

## §17. SUBGROUP

A group which is contained in another group is called its **subgroup**, and we denote a group and its subgroup by $G$ and $H$ respectively when we do not use special notation.

Such groups stand in a beautiful relation. Let the permutations of the set

$$H = 1, s_2, \ldots, s_r$$

form a subgroup of order $r$ in $G$ and let $\psi$ be a function that belongs to $H$. If $t$ is a permutation contained in $G$ but not in $H$ and converts $\psi$ into $\psi_t$ such that

$$\psi_t \neq \psi,$$

[1] Cf. §22; also §8.

then every permutation of the set

$$Ht = t,\ s_2t,\ \ldots,\ s_rt$$

converts $\psi$ into $\psi_t$, for $s_i$ leaves $\psi$ unaltered and $t$ effects the change.

There is no permutation outside $Ht$ that converts $\psi$ into $\psi_t$. Suppose that $\tau$ does so; then $\tau t^{-1}$ leaves $\psi$ unaltered as $t^{-1}$ converts $\psi_t$ into $\psi$. Hence

$$\tau t^{-1} = s_i,$$

and after multiplication by $t$ on the right:

$$\tau = s_i\, t.$$

The permutations of $Ht$ are all different from the permutations of $H$, and they are all unlike since from

$$s_i\, t = s_k\, t$$

would follow

$$s_i = s_k.$$

If the sets of $H$ and $Ht$ do not contain all the permutations of $G$, there is a permutation $u$ in $G$ but neither in $H$ nor in $Ht$ that converts $\psi$ into $\psi_u$. And so does every permutation of the set

$$Hu = u,\ s_2u,\ \ldots,\ s_ru,$$

to which all conclusions for $Ht$ apply.

Continuing so until the group $G$ is exhausted:

$$\begin{array}{ll} H = 1,\ s_2,\ \ldots,\ s_r & [\psi \rightarrow \psi \\ Ht = t,\ s_2t,\ \ldots,\ s_rt & [\psi \rightarrow \psi_t \\ Hu = u,\ s_2u,\ \ldots,\ s_ru & [\psi \rightarrow \psi_u \\ \ldots\ldots\ldots\ldots, & \end{array}$$

we arrange the permutations of

$$G = H + Ht + Hu + \ldots$$

in **partitions** of $G$ with respect to $H$ or **co-sets** of $H$ in $G$, as we say; and they convert $\psi$ into

$$\psi,\ \psi_t,\ \psi_u,\ \ldots$$

respectively, which functions are called **conjugate** with $\psi$ under $G$.

We may arrange the permutations so as to set

$$G = H + tH + uH + \ldots,$$

but in general

$$Ht \neq tH,$$

for the commutative law does not always hold for permutations and sets of such.[1]

Since every partition contains as many permutations as $H$ does, and all partitions together contain as many permutations as $G$ does, the proposition follows which is due to Lagrange:

(23) **The order of a group is divisible by the order of any subgroup.**

If $r_g$ denotes the order of $G$ and $r_h$ the order of $H$, then

$$r_g = j \,.\, r_h,$$

where $j$ is called the **index** of $H$ in $G$. It gives the number of partitions in $G$ with respect to $H$ and the number of conjugate values that a function $\psi$ belonging to $H$ takes under $G$.

Any group on the $n$ letters $x_i$ is contained in the symmetric group $S$ on these letters. Denoting the index of $G$ in $S$ by $j_g$ and the index of $H$ in $S$ by $j_h$, we have therefore

$$n! = j_g r_g = j_h r_h,$$

so that the index $j$ or, preciser, $j_{hg}$ of $H$ in $G$ is expressible as a quotient of either orders or indices:

$$j_{hg} = \frac{r_g}{r_h} = \frac{j_h}{j_g}.$$

Since the order of any possible subgroup $H$ of $S$ has to divide $n!$, there are for $n = 4$, let us say, no other subgroups of $S$ possible than such as have

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $r_h =$ | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 24 |
| $j_h =$ | 24 | 12 | 8 | 6 | 4 | 3 | 2 | 1. |

As illustration may serve an example:

$H \;\; = 1, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)$

$$\psi = x_1x_2 + x_3x_4$$

$Ht \; = (23), (132), (234), (1342), (1243), (14), (124), (143)$

$$\psi_t = x_1x_3 + x_2x_4 \qquad [t = (23)$$

$Hu = (24), (142), (243), (1432), (13), (1234), (134), (123)$

$$\psi_u = x_1x_4 + x_2x_3. \qquad [u = (24)$$

[1] By §11. But cf. proposition (32): $Nt = tN$; cf. also proposition (82).

In a sense every group $G$ contains itself and identity as subgroups, but it often is more convenient to consider one or both as not included among them. If this is not specified, it will be clear from the context.

Since the powers of a permutation $s$ in $G$ by themselves compose a cyclic group $\{s\}$ and this group is a subgroup of $G$,

(24) **the order of any permutation in a group divides the order of the group.**

If the order of a group $G$ is prime, it follows that the powers of a permutation $s$ in $G$ compose the whole group which then is cyclic:

$$G = \{s\};$$

and since $s$ is any permutation in $G$, that

$$\{s\} = \{s^k\},$$

which becomes intelligible if we recall that by proposition (16) the powers of $s$ are similar permutations: the order of $s^k$ equals the order of $s$ while every power of $s^k$ equals some power of $s$.

Calling a cyclic group $\{s\}$ **circular** when $s$ is so, we note that

(25) **a group of prime order is cyclic, a group of prime order and degree is circular;**

the latter by proposition (17).

## §18. CONJUGATE SUBGROUPS

While the permutations

$$Ht = t,\ s_2\,t,\ .\ .\ .\ ,\ s_r\,t$$

of $G$ convert the function $\psi$ belonging to

$$H = 1,\ s_2,\ .\ .\ .\ ,\ s_r$$

into the conjugate function $\psi_t$ and compose a partition of $G$, they do not compose the group of $\psi_t$; not even a group at all since we notice that they do not contain identity and contain $t$ but none of its powers.

To find the group of $\psi_t$ which we denote by $H_t$, we let $\tau$ be any permutation contained in it, so that $\tau$ applied to $\psi_t$ leaves it unaltered:

$$\psi_{t\tau} = \psi_t.$$

To both sides of this identity we apply the permutation $t^{-1}$ and obtain

$$\psi_{t\tau t^{-1}} = \psi_{tt^{-1}} = \psi$$

since the permutation $tt^{-1} = 1$ does not alter $\psi$. It follows that the permutation $t\tau t^{-1}$ does not alter $\psi$ either and is some permutation of $H$:

$$t\tau t^{-1} = s_i.$$

Hence we find

$$\tau = t^{-1}s_i\, t$$

as shown in example (d) of §15.

The permutation $t^{-1}s_i\, t$ is called the **transform of permutation** $s_i$ by $t$. Any such transform leaves $\psi_t$ unaltered as $t^{-1}$ changes $\psi_t$ into $\psi$, and $s_i$ leaves $\psi$ fixed, and $t$ changes $\psi$ back into $\psi_t$. Since any permutation $\tau$ of $H_t$ is the transform of a permutation $s_i$ in $H$ by $t$, and any transform of a permutation $s_i$ in $H$ by $t$ leaves $\psi_t$ unaltered, we infer that the group of $\psi_t$ is

$$H_t = 1,\ t^{-1}s_2\, t,\ \ldots,\ t^{-1}s_r\, t. \qquad [t^{-1}s_1\, t = 1$$

This set of transforms of the permutations in $H$ by $t$ is called the **transform of group** $H$ by $t$ and denoted[1] by $t^{-1}Ht$; hence we can set

$$H_t = t^{-1}Ht.$$

It is clear that $H_t$ is a subgroup of $G$. Since the permutations of $H$ are all distinct, their transforms are so, too, and $H_t$ must have the same order and index in $G$ that $H$ has.

As the function $\psi_t$ is called conjugate with $\psi$ under $G$, so the permutation $\tau$ is called **conjugate** with $s_i$ under $G$, and the subgroup $H_t$ is called **conjugate** with $H$ under $G$.

It follows that

(26) **the transform of a permutation is a conjugate permutation, the transform of a subgroup is a conjugate subgroup,**

and that

(27) **conjugate subgroups have the same order and index.**

Corresponding to the partitions

$$H,\ Ht,\ Hu,\ \ldots$$

[1] Sometimes by $tHt^{-1}$, which is read from the right.

and the conjugate functions

$$\psi, \psi_t, \psi_u, \ldots$$

we thus have the conjugate subgroups

$$H, H_t, H_u, \ldots$$

such that

$$H_t = t^{-1}Ht$$
$$H_u = u^{-1}Hu$$
$$\cdots\cdots\cdots$$

This gives the proposition:

(28) **If $j$ is the index of $H$ in $G$, a function $\psi$ belonging to $H$ takes $j$ conjugate values under $G$ which belong to subgroups conjugate with $H$ under $G$.**

Since the product of any two permutations $s_i$ and $s_j$ of $H$ is equal to a permutation $s_k$ contained in $H$, the product of their transforms by any permutation $t$ of $G$ is

$$t^{-1}s_i\, t \,.\, t^{-1}s_j\, t = t^{-1}s_i\, s_j\, t = t^{-1}s_k\, t$$

contained in $t^{-1}Ht$. This verifies that the transform of $H$ by $t$ is a group.

The transform of a product equals the product of the transforms, for

$$t^{-1}s_i\, s_j\, t = t^{-1}s_i\, t \,.\, t^{-1}s_j\, t,$$

whence the transform of a power equals that power of the transform:

$$t^{-1}s_i^2\, t = (t^{-1}s_i\, t)^2.$$

The transform of the inverse equals the inverse of the transform:

$$t^{-1}s_i^{-1}\, t = (t^{-1}s_i\, t)^{-1},$$

for

$$t^{-1}s_i^{-1}\, t \,.\, t^{-1}s_i\, t = 1.$$

## §19. RULE OF TRANSFORMS

To avoid the multiplication of three permutations in computing transforms, we make use of a simple device.

Suppose we have the permutations

$$s = (abcd)$$

and

$$t = \begin{pmatrix} abcd \ldots \\ klmn \ldots \end{pmatrix},$$

the letters in parenthesis denoting the subscripts of the $x_i$ upon which the permutations $s$ and $t$ operate. The inverse of $t$ is

$$t^{-1} = \begin{pmatrix} klmn \ . \ . \ . \\ abcd \ . \ . \ . \end{pmatrix},$$

and the transform $t^{-1}s\,t$ replaces $k$ by $a$ by $b$ by $l$, $l$ by $b$ by $c$ by $m$, $m$ by $c$ by $d$ by $n$, $n$ by $d$ by $a$ by $k$, which closes the cycle:

$$t^{-1}s\,t = (klmn).$$

But this result we obtain simpler if we replace the subscripts in the cycle of the permutation $s$ as indicated by the permutation $t$.

When $s$ is not a circular permutation, we represent it by proposition (13) as product of circular permutations $s_1$ and $s_2$, say, decomposing it into cycles. We then perform the operation on every cycle, which we can do since

$$t^{-1}s\,t = t^{-1}s_1s_2\,t = t^{-1}s_1\,t \,.\, t^{-1}s_2\,t.$$

Hence we have the **rule of transforms**:

(29) **The transform of a permutation $s$ by a permutation $t$ is obtained by operating $t$ within the cycles of $s$;**

and it is clear that

(30) **the transform of a permutation $s$ is a permutation similar to $s$.**

The order of any transform of a permutation is therefore the same as that of the permutation itself.

An example may illustrate the rule:

$H$ = 1, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)

$$\psi = x_1x_2 + x_3x_4$$

$H_t$ = 1, (13), (24), (13)(24), (12)(34), (14)(32), (1234), (1432)

$$\psi_t = x_1x_3 + x_2x_4 \qquad [t = (23)$$

$H_u$ = 1, (14), (32), (14)(32), (13)(42), (12)(43), (1342), (1243)

$$\psi_u = x_1x_4 + x_3x_2. \qquad [u = (24)$$

This example makes the rule of transforms self-evident: if we convert $\psi$ into $\psi_t$ by interchanging $x_2$ and $x_3$, we have only to interchange $x_2$ and $x_3$ in the permutations of the group to which $\psi$ belongs in order to obtain the group of $\psi_t$.

It will be noticed that the permutations of conjugate subgroups do not make up a group:

$$G = H + Ht + Hu + \ . \ . \ .$$

but

$$G \neq H + H_t + H_u + \dots$$

This may be verified by taking as $G$ the symmetric group on four letters given in §21 and as $H$ the group above.

### §20. NORMAL SUBGROUP

If a subgroup of $G$ is identical with the subgroups conjugate to it under $G$, so that transforming it by any permutation of $G$ we reproduce the same subgroup, the subgroup is called **normal or invariant or self-conjugate**[1] in $G$ and is denoted by $N$ or $J$.

If $N$ is a normal subgroup of $G$, we have therefore

$$t^{-1}Nt = N$$

'or any permutation $t$ of $G$, whence $N$ contains with a permutation ; also every transform of $s$ by a permutation of $G$. These trans- 'orms are by proposition (30) permutations similar to $s$; yet $N$ may contain not all permutations of $G$ similar to $s$. For instance, the group

$$G = 1, (12), (34), (12)(34)$$

has a normal subgroup

$$N = 1, (12)$$

not containing the permutation (34) similar to (12).

But every permutation similar to $s$ is its transform by some other permutation, and that permutation is by necessity contained in the symmetric group; consequently

(31) **a normal subgroup of the symmetric group contains with a permutation *s* also every permutation of that group which is similar to *s*,**

therefore every possible such permutation.

From the relation

$$t^{-1}Nt = N$$

we obtain the other:[2]

$$Nt = tN,$$

[1] Normal subgroups were discovered by Galois.

[2] Cf. §17, where $Ht \neq tH$. Cf. also proposition (82).

and conversely. This means that

(32) **every permutation of a group is commutative with a subgroup when, and only when, the subgroup is normal.**

A subgroup which is normal in the group $G$ and contained in the subgroup $H$ of $G$ is evidently normal also in $H$, but a subgroup which is normal in $H$ is not necessarily normal in $G$. Thus the group $N$ of the example above is not normal in the symmetric group.[1]

Every group $G$ may be taken to contain itself and identity as normal subgroups, because for any permutation $t$ of $G$ we have

$$t^{-1}1t = 1$$

and

$$t^{-1}Gt = G.$$

But we do not ordinarily mean these groups when we speak of normal subgroups.

A group containing no normal subgroup except itself and identity is called **simple**; a group which is not simple is **composite.**

A group of prime order always is simple, since it can have no subgroups whatever different from itself and identity.

Normal subgroups of their symmetric groups are, for instance:

(a) $N = 1,\ (123),\ (132)$

$$\psi = (x_1 + \omega x_2 + \omega^2 x_3)^3$$

$N_t = 1,\ (132),\ (123) = N$

$$\psi_t = (x_1 + \omega^2 x_2 + \omega x_3)^3. \qquad [t = (23)$$

(b) $N = 1,\ (12)(34),\ (13)(24),\ (14)(23)$

$$\psi = (x_1 - x_2)(x_3 - x_4)$$

$N_t = 1,\ (13)(42),\ (14)(32),\ (12)(34) = N$

$$\psi_t = (x_1 - x_3)(x_4 - x_2) \qquad [t = (234)$$

$N_u = 1,\ (14)(23),\ (12)(43),\ (13)(42) = N$

$$\psi_u = (x_1 - x_4)(x_2 - x_3). \qquad [u = (243)$$

[1] Another example is in §46.

# CHAPTER V

## SYMMETRIC GROUP AND ITS FUNCTIONS

### §21. GENERATOR

The permutations of a set are called **independent** if no one of them can be expressed as a product of the rest. They define a group and as such they are called **generating permutations.** To obtain that group, we combine the generators in all possible ways until we get only such permutations as we already have.

The $n - 1$ independent transpositions

$$(x_1x_i)_2^n = (x_1x_2),\ (x_1x_3),\ .\ .\ .\ ,\ (x_1x_n)$$

generate the symmetric group on the $n$ letters $x_i$. To verify this, we consider that every permutation is composed of cycles or resolvable into such by proposition (13); that every cycle is decomposable into transpositions:

$$(x_1x_2x_3x_4) = (x_1x_2)(x_1x_3)(x_1x_4);$$

and that every transposition is expressible as product of some transpositions in $(x_1x_n)_2^n$:

$$(x_2x_3) = (x_1x_2)(x_1x_3)(x_1x_2).$$

Since there is no permutation in the symmetric group which is not a combination of transpositions in $(x_1x_i)_2^n$, we infer that all possible combinations of these transpositions give us the symmetric group.

If $\{s,t\}$ stands for the group generated by the permutations $s$ and $t$, we can note therefore that

(33) **the symmetric group on $n$ letters $x_i$ is generated by the $n - 1$ independent transpositions $(x_1x_i)_2^n$:**

$$\boxed{S^n = \{(x_1x_i)_2^n\}}.$$

It follows that a group is symmetric whenever it contains all transpositions $(x_1x_i)_2^n$.

For the lowest degrees the symmetric groups are:

$S^2 = 1, (12)$

$S^3 =$

| | | |
|---|---|---|
| 1 | (13) | (23) |
| (12) | (123) | (132) |

$S^4 =$

| | | | |
|---|---|---|---|
| 1 | (14) | (24) | (34) |
| (12)<br>(13)<br>(23)<br>(123)<br>(132) | (124)<br>(134)<br>(14)(23)<br>(1234)<br>(1324) | (142)<br>(13)(24)<br>(234)<br>(1423)<br>(1342) | (12)(34)<br>(143)<br>(243)<br>(1243)<br>(1432) |

As to the rule by which to form a symmetric group of higher degree from the preceding symmetric group of lower degree, we notice that

$$S^3 = S^2 + S^2t_1 + S^2t_2,$$

where

$$t_1 = (13) \text{ or } (123)$$
$$t_2 = (23) \text{ or } (132);$$

and

$$S^4 = S^3 + S^3t_1 + S^3t_2 + S^3t_3,$$

where

$$t_1 = (14) \text{ or } (1234)$$
$$t_2 = (24) \text{ or } (13)(24)$$
$$t_3 = (34) \text{ or } (1432).$$

In general

$$S^n = S^{n-1} + S^{n-1}t_1 + S^{n-1}t_2 + \cdots + S^{n-1}t_{n-1},$$

where

$$t_1 = (1n) \text{ or } (1 \ldots n)$$
$$t_2 = (2n) \text{ or } (1 \ldots n)^2$$
$$\cdots\cdots\cdots$$
$$t_{n-1} = (n-1\ n) \text{ or } (1 \ldots n)^{n-1}.$$

## §22. SYMMETRIC SUM

Any function of the $n$ letters $x_i$ that remains unaltered by the permutations of the symmetric group on these letters is called a **symmetric function** of the $x_i$ and denoted by $S$. It remains unaltered because the permutations of the symmetric group do nothing but interchange its terms.[1]

A symmetric function does not have to be homogeneous in the $x_i$. But if it is not, we evidently can express it as a sum of homogeneous functions collecting the terms of one total degree[2] in the $x_i$ into one such function.

Every homogeneous function which is part of a symmetric function is symmetric itself because permutations, interchanging the terms of the symmetric function, interchange the terms within every homogeneous function only, since they cannot alter the total degree of a term.

If a homogeneous symmetric function contains the term

$$a \,.\, x_1^{\mu_1} x_2^{\mu_2} \;.\;.\;.\; x_k^{\mu_k},$$

it also contains every term that is obtained applying to this the permutations of the symmetric group which interchange the subscripts of the $x_i$ leaving the exponents fixed.

The sum of terms obtained in such a manner we call a **symmetric sum** and denote by $s$. With the term that we noted the homogeneous symmetric function contains the symmetric sum

$$s(a \,.\, x_1^{\mu_1} \;.\;.\;.\; x_k^{\mu_k}) = a \,.\, s(x_1^{\mu_1} \;.\;.\;.\; x_k^{\mu_k});$$

the notation will be readily understood. This symmetric sum has $k!$ terms if all the exponents $\mu_i$ are distinct; if some of them are alike, including those that may be zero, the number of terms is a fraction only of $k!$.

A term of the homogeneous symmetric function which is not contained in this symmetric sum belongs to another, so that a homogeneous symmetric function is composed of symmetric sums. They are all of the same total degree in any one term since they belong to one homogeneous function.

[1] This applies to integral symmetric functions; fractional symmetric functions are expressible as quotients of integral symmetric functions, which alone we consider.

[2] Cf. §24.

## §23. COMPUTATION OF SYMMETRIC SUM

We prove that

(34) **a rational symmetric sum in the $n$ letters $x_i$ which represent the roots of the general equation**

$$\boxed{f(x) = x^n - c_1x^{n-1} + \ldots \pm c_n = 0}$$

**is rationally expressible in terms of the elementary symmetric functions**

$$\boxed{\begin{array}{ll} s(x_1) & = x_1 + x_2 + x_3 + \ldots = c_1 \\ s(x_1x_2) & = x_1x_2 + x_1x_3 + \ldots = c_2 \\ & \ldots\ldots\ldots \\ s(x_1 \ldots x_n) & = x_1x_2x_3 \ldots x_n = c_n \end{array}}.$$

To obtain a unique arrangement of terms in any function, we agree to call the first of two terms

$$p \,.\, x_1^{\mu_1}x_2^{\mu_2} \ldots x_k^{\mu_k}$$

and

$$q \,.\, x_1^{\nu_1}x_2^{\nu_2} \ldots x_l^{\nu_l}$$

higher if $\mu_1 > \nu_1$ or, $\mu_1$ being equal to $\nu_1$, if $\mu_2 > \nu_2$, and so on. This convention will hold also in the case that some $x_i$ has the exponent zero and disappears.

To obtain a unique arrangement of functions, like symmetric sums, we agree to call the one higher whose highest term is higher.

Two terms could be equally high only if all corresponding exponents were equal, but would for that same reason be added into one term. Likewise, two equally high symmetric sums would be added into one sum.

The highest terms of the elementary symmetric functions

$$s(x_1),\ s(x_1x_2),\ s(x_1x_2x_3),\ \ldots$$

are respectively

$$x_1, \quad x_1x_2, \quad x_1x_2x_3, \ \ldots$$

The highest term of the homogeneous symmetric function

$$c_1^{\mu_1-\mu_2}c_2^{\mu_2-\mu_3} \ldots c_{k-1}^{\mu_{k-1}-\mu_k}c_k^{\mu_k} = C$$

of the $x_i$ is obtained as the product of the highest terms in its factors if we assume that their exponents are not negative:

$$\mu_1 - \mu_2 \geqq 0,\ \mu_2 - \mu_3 \geqq 0,\ \ldots$$

It is

$$x_1^{\mu_1-\mu_2}(x_1x_2)^{\mu_2-\mu_3} \ldots (x_1 \ldots x_k)^{\mu_k} = x_1^{\mu_1}x_2^{\mu_2} \ldots x_k^{\mu_k},$$

so that the highest symmetric sum in $C$ is

$$s(x_1^{\mu_1}x_2^{\mu_2} \ldots x_k^{\mu_k}).$$

It follows that $x_1^{\mu_1} \ldots x_k^{\mu_k}$, being the highest term in $C$, also is the highest term in $s(x_1^{\mu_1} \ldots x_k^{\mu_k})$, with which we readily agree recalling the assumption as to the exponents and writing it in the form

$$\mu_1 \geqq \mu_2 \geqq \ldots \geqq \mu_k.$$

With this assumption made, we prove our proposition for $s(x_1^{\mu_1}x_2^{\mu_2} \ldots x_k^{\mu_k})$ on the lines of actual computation.

Subtracting $s$ from $C$ we obtain

$$C - s = S,$$

where

$$S = \sum_1^m p_\iota s_\iota\,(x_1^{\mu_{\iota 1}}\,x_2^{\mu_{\iota 2}} \ldots x_{k_\iota}^{\mu_{\iota k_\iota}})$$

is a homogeneous symmetric function composed of symmetric sums $s_i$ all lower than $s$ and arranged so that every $s_i$ is higher than $s_{i+1}$ while every

$$\mu_{ij} \geqq \mu_{ij+1}.$$

For the lowest possible sum $s_m$ with

$$\mu_{m1} = \mu_{m2} = \ldots = \mu_{mk_m} = 1$$

and

$$k_m = n$$

we have the total degree

$$\sum \mu_i = \sum \mu_{ij} = n.$$

From $S$ we subtract

$$p_1C_1 = p_1c_1^{\mu_{11}-\mu_{12}} \ldots c_{k_1}^{\mu_{1k_1}}$$

with the highest term $p_1x_1^{\mu_{11}} \ldots x_{k_1}^{\mu_{1k_1}}$ and the highest symmetric sum

$$p_1s_1(x_1^{\mu_{11}} \ldots x_{k_1}^{\mu_{1k_1}}).$$

This removes $p_1s_1$ from $S$ and gives

$$S - p_1C_1 = S_1,$$

where

$$S_1 = \sum_{2}^{m} q_\iota s_\iota \,(x_1^{\mu_{\iota 1}} \ . \ . \ . \ x_{k_\iota}^{\mu_{\iota k_\iota}})$$

is composed of symmetric sums all lower than $s_1$. With $S_1$ we now proceed as we did with $S$.

Having eliminated successively the highest remaining symmetric sums, we rewrite our equations of elimination

$$\begin{aligned} s &= C - S \\ 0 &= S - p_1 C_1 - S_1 \\ &\ . \ . \ . \ . \ . \ . \ . \end{aligned}$$

and adding them obtain

$$s = C - p_1 C_1 \pm q_2 C_2 \pm \ . \ . \ . \ ,$$

which proves our proposition.

Since any symmetric function is composed of homogeneous symmetric functions, any homogeneous symmetric function is composed of symmetric sums, and any symmetric sum is rationally expressible in terms of the elementary symmetric functions-we have the **fundamental theorem of symmetric functions:**

(35) **Every rational symmetric function is rationally expressible in terms of the elementary symmetric functions.**

While the procedure of the proof may seem complicated, it will become clear from examples:

(a) $s(x_1^2 x_2^2)$.

$$\begin{aligned} S &= c_2^2 - s(x_1^2x_2^2) = (x_1x_2 + \ . \ . \ . \ + x_3x_4)^2 - s(x_1^2x_2^2) \\ &= [s(x_1^2x_2^2) + 2s_1(x_1^2x_2x_3) + 6s_2(x_1x_2x_3x_4)] - s(x_1^2x_2^2) \\ &= 2s_1(x_1^2x_2x_3) + 6s_2(x_1x_2x_3x_4) \\ S_1 &= S - 2c_1c_3 = S - 2[s_1(x_1^2x_2x_3) + 4s_2(x_1x_2x_3x_4)] \\ &= 2s_2(x_1x_2x_3x_4) = 2c_4 \\ s &= c_2^2 - 2c_1c_3 + 2c_4. \end{aligned}$$

To expand

$$c_2^2 = (x_1x_2 + \ . \ . \ . \ + x_3x_4)^2,$$

we observe that the term $x_1^2x_2^2$ can appear only once, as the product $x_1x_2 \,.\, x_1x_2$, this determining the coefficient of $s$ in the expansion. The term $x_1^2x_2x_3$ twice, as product $x_1x_2 \,.\, x_1x_3$ and $x_1x_3 \,.\, x_1x_2$, which gives the coefficient of $s_1$. The term $x_1x_2x_3x_4$ six

times, as product $x_1x_2 \,.\, x_3x_4$, $x_1x_3 \,.\, x_2x_4$, $x_1x_4 \,.\, x_2x_3$, $x_2x_3 \,.\, x_1x_4$, $x_2x_4 \,.\, x_1x_3$, $x_3x_4 \,.\, x_1x_2$, which gives the coefficient of $s_2$.

To expand

$$c_1c_3 = (x_1 + \ . \ . \ . \ +x_4)(x_1x_2x_3 + \ . \ . \ . \ +x_2x_3x_4),$$

we notice that the term $x_1{}^2x_2x_3$ appears once, as the product $x_1 \cdot x_1x_2x_3$, this determining the coefficient of $s_1$; the term $x_1x_2x_3x_4$ four times, as product $x_1 \,.\, x_2x_3x_4$, $x_2 \,.\, x_1x_3x_4$, $x_3 \,.\, x_1x_2x_4$, $x_4 \,.\, x_1x_2x_3$, which gives the coefficient of $s_2$.

(b) $s(x_1{}^2x_2{}^2x_3)$.

$$\begin{aligned} S &= c_2c_3 - s = 3s_1(x_1{}^2x_2x_3x_4) + 10\ s_2(x_1x_2x_3x_4x_5) \\ S_1 &= S - 3c_1c_4 = 5s_2(x_1x_2x_3x_4x_5) = 5c_5 \\ s &= c_2c_3 - 3c_1c_4 + 5c_5. \end{aligned}$$

Since the highest subscript of the $c_i$ determines the degree of the general equation whose coefficients they are and the number of its roots $x_i$, the highest subscript of the $c_i$ may not be greater than the number of the $x_i$ in the symmetric sum $s$, and higher subscripts must not appear in the computation. Marking the number of the $x_i$ in $s$ by an index in parenthesis, we have for instance:

(a) $s^{(2)}(x_1{}^2x_2{}^2) = c_2{}^2$; $s^{(3)}(x_1{}^2x_2{}^2) = c_2{}^2 - 2c_1c_3$;
$s^{(4)}(x_1{}^2x_2{}^2) = c_2{}^2 - 2c_1c_3 + 2c_4$.

(b) $s^{(3)}(x_1{}^2x_2{}^2x_3) = c_2c_3$; $s^{(4)}(x_1{}^2x_2{}^2x_3) = c_2c_3 - 3c_1c_4$;
$s^{(5)}(x_1{}^2x_2{}^2x_3) = c_2c_3 - 3c_1c_4 + 5c_5$.

## §24. ANOTHER COMPUTATION

To learn how to avoid calculations like those which precede, we set

$$s(x_1{}^{\mu_1} \ . \ . \ . \ x_k{}^{\mu_k}) = \sum p_\nu c_1{}^{\nu_1}c_2{}^{\nu_2} \ . \ . \ . \ c_n{}^{\nu_n},$$

which we now know to be true. Replacing every $x_i$ by $\lambda x_i$, we change $c_1$ into $\lambda c_1$, $c_2$ into $\lambda^2 c_2$, . . . , the exponent of $\lambda$ depending on the degree of $c_i$ in the $x_i$, and we obtain

$$\lambda^{\mu_1 + \ . \ . \ . \ + \mu_k} s(x_1{}^{\mu_1} \ . \ . \ . \ x_k{}^{\mu_k}) = \sum \lambda^{\nu_1 + 2\nu_2 + \ . \ . \ . \ + n\nu_n} p_\nu c_1{}^{\nu_1}c_2{}^{\nu_2} \ . \ . \ . \ c_n{}^{\nu_n}.$$

Hence we see that for every term of $\sum$ holds the relation

$$\nu_1 + 2\nu_2 + \ . \ . \ . \ + n\nu_n = \mu_1 + \ . \ . \ . \ + \mu_k.$$

The product of exponent and subscript of a letter is called the **weight of a letter.** The sum of the weights of letters in a term is called the **weight of a term** in those letters and is denoted by $W$. The sum of the degrees of letters in a term is called the **total degree** of a term in those letters and is denoted by $D$.

Since the weight in the $c_i$ of all terms in $\sum$ thus is constant and equal to the total degree of $s$ in the $x_i$, we can express $s$ in terms of the $c_i$ by simply restricting our choice of combinations $c_1^{\nu_1} \ldots c_n^{\nu_n}$ to such as satisfy the condition of weight. If then $\mu_1$ is the highest degree of any $x_i$ in a term of $s$, the total degree of the $c_i$ in any term of $\sum$ must not exceed $\mu_1$, which may exclude some combinations of the $c_i$ that pass the restriction of weight.

Applying this we find:

$$\text{(a)}\quad s(x_1^2x_2^2) = \sum p_\nu c_1^{\nu_1} \ldots c_n^{\nu_n}.$$

$$W = 4 \text{ permits } c_1^4,\ c_2^2,\ c_1c_3,\ c_4$$

$$D \leqq 2 \text{ excludes } c_1^4$$

$$s(x_1^2x_2^2) = c_2^2 + pc_1c_3 + qc_4.$$

$$\text{(b)}\quad s(x_1^2x_2^2x_3) = \sum p_\nu c_1^{\nu_1} \ldots c_n^{\nu_n}.$$

$$W = 5 \text{ permits } c_1c_2^2,\ c_2c_3,\ c_1c_4,\ c_5$$

$$D \leqq 2 \text{ excludes } c_1c_2^2$$

$$s(x_1^2x_2^2x_3) = c_2c_3 + pc_1c_4 + qc_5.$$

For computation of the numerical coefficients, which remains to be done and which is done best by the use of special equations, we refer to the examples of chapter eight.

Also, there exist tables with the results of such computation. They are arranged according to the total degree of the symmetric sums, and we reproduce the table for the total degree four:

| | $c_4$ | $c_3c_1$ | $c_2^2$ | $c_2c_1^2$ | $c_{14}$ |
|---|---|---|---|---|---|
| $s(x_1^4)$ | $-4$ | 4 | 2 | $-4$ | 1 |
| $s(x_1^3x_2)$ | 4 | $-1$ | $-2$ | 1 | |
| $s(x_1^2x_2^2)$ | 2 | $-2$ | 1 | | |
| $s(x_1^2x_2x_3)$ | $-4$ | 1 | | | |
| $s(x_1x_2x_3x_4)$ | 1 | | | | |

This table gives, for instance:

$$s^{(4)}(x_1^4) = c_1^4 - 4c_1^2c_2 + 2c_2^2 + 4c_1c_3 - 4c_4$$
$$s^{(2)}(x_1^4) = c_1^4 - 4c_1^2c_2 + 2c_2^2.$$

Since symmetric functions are composed of symmetric sums, the rules of computation may be applied to them directly, the highest weight and degree setting the mark.

## §25. RESULTANT[1]

Among the symmetric functions in the letters $x_i$ which represent the roots of the general equation

$$f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m = 0$$

are the resultant and the discriminant.

If besides the function $f(x)$ we have another such function

$$g(x) = b_0x^n + b_1x^{n-1} + \cdots + b_n,$$

it will be convenient to denote the roots of $f(x)$ by $\alpha_i$ and the roots of $g(x)$ by $\beta_i$.

The two functions $f(x)$ and $g(x)$ have a root $\alpha_i$ in common if the product

$$g(\alpha_1)g(\alpha_2) \cdots g(\alpha_m)$$

vanishes with $g(\alpha_i)$ becoming zero. This product is evidently an integral function in the $b_i$. Since a permutation between the $\alpha_i$ only interchanges its factors, the product is a symmetric function of the $\alpha_i$ and as such by proposition (35) rationally expressible in terms of

$$\frac{a_1}{a_0}, \frac{a_2}{a_0}, \ldots, \frac{a_m}{a_0}.$$

Since the product is of degree not more than $n$ in any one $\alpha_i$, it is of total degree not more than $n$ in the $a_i/a_0$. Multiplying it by $a_0^n$, we therefore obtain a function

$$R(f,g) = a_0^ng(\alpha_1)g(\alpha_2) \cdots g(\alpha_m)$$

which is integral in the $a_i$ as it is in the $b_i$, and this function is called the **resultant** of $f(x)$ and $g(x)$.

[1] The rest of this chapter may be omitted on first reading.

It appears that  of the

(36) **the vanishing resultant of two functions indicates that the two functions have a root in common.**

Since

$$g(x) = b_0(x - \beta_1)(x - \beta_2)\ .\ .\ .\ (x - \beta_n)$$

and hence

$$g(\alpha_1) = b_0(\alpha_1 - \beta_1)(\alpha_1 - \beta_2)\ .\ .\ .\ (\alpha_1 - \beta_n)$$
$$g(\alpha_2) = b_0(\alpha_2 - \beta_1)(\alpha_2 - \beta_2)\ .\ .\ .\ (\alpha_2 - \beta_n)$$
$$.\ .\ .\ .\ .\ .\ .\ .\ ,$$

the resultant takes the form

$$R(f,g) = a_0{}^n b_0{}^m \prod_{i,\,k}(\alpha_i - \beta_k), \quad [i = 1, \ldots, m;\ k = 1, \ldots, n$$

and a mere glance suffices to verify that its vanishing is a necessary and sufficient condition for any common root.

Interchanging the two functions $f(x)$ and $g(x)$ we may or may not alter the sign of the resultant, for

$$\begin{aligned} R(f,g) &= (-1)^{mn}R(g,f) \\ &= (-1)^{mn}b^m f(\beta_1)f(\beta_2)\ .\ .\ .\ f(\beta_n). \end{aligned}$$

## §26. RESULTANT AS DETERMINANT

To express the resultant of the two functions $f(x)$ and $g(x)$ in terms of their coefficients, we search for it anew. We already know that it is an integral function of the coefficients $a_i$ and $b_i$.

Suppose that $\alpha_i$ is a common root of the two functions. Then we may set

$$f(x) = (x - \alpha_i)\ .\ f_1(x)$$

and

$$g(x) = (x - \alpha_i)\ .\ g_1(x),$$

where $f_1(x)$ is of degree $m - 1$ and $g_1(x)$ of degree $n - 1$ and both are integral functions of $x$. Eliminating $x - \alpha_i$ from the two equations, we find

$$f(x)\ .\ g_1(x) - f_1(x)\ .\ g(x) = 0.$$

Assuming that

$$f_1(x) = p_0x^{m-1} + p_1x^{m-2} + \ .\ .\ .\ + p_{m-2}x + p_{m-1}$$
$$g_1(x) = q_0x^{n-1} + q_1x^{n-2} + \ .\ .\ .\ + q_{n-2}x + q_{n-1}$$

and substituting, we find

$$(a_0q_0 - b_0p_0)x^{m+n-1} + (a_1q_0 + a_0q_1 - b_1p_0 - b_0p_1)x^{m+n-2} + \dots$$
$$+ (a_mq_{n-2} + a_{m-1}q_{n-1} - b_np_{m-2} - b_{n-1}p_{m-1})x$$
$$+ (a_mq_{n-1} - b_np_{m-1}) = 0.$$

This being an identity, the coefficients of $x$ must be zero. whence $p$ and $q$ must satisfy the equations

$$\begin{array}{llr}
a_0q_0 & -b_0p_0 & = 0 \\
a_1q_0 + a_0q_1 & - b_1p_0 - b_0p_1 & = 0 \\
\dots & \dots & \\
a_mq_{n-2} + a_{m-1}q_{n-1} & -b_np_{m-2} - b_{n-1}p_{m-1} & = 0 \\
a_mq_{n-1} & - b_np_{m-1} & = 0
\end{array}$$

They will satisfy these equations if the determinant of their coefficients vanishes. With rows and columns interchanged, the determinant is

$$R_d = \left|\begin{array}{cccccccc}
a_0 & a_1 & \dots & a_m & 0 & \dots & 0 \\
0 & a_0 & \dots & a_{m-1} & a_m & \dots & 0 \\
 & & \dots & \dots & \dots & & \\
0 & 0 & \dots & \dots & \dots & \dots & a_m \\
b_0 & b_1 & \dots & \dots & \dots & \dots & 0 \\
0 & b_0 & \dots & \dots & \dots & \dots & 0 \\
 & & \dots & \dots & \dots & & \\
0 & 0 & \dots & \dots & \dots & \dots & b_n
\end{array}\right| \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} n \text{ rows} \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} m \text{ rows}\end{array}$$

This determinant is obviously homogeneous of total degree $n$ in the $a_i$ and of total degree $m$ in the $b_i$, so that its terms are of the type

$$k \cdot a_0^{\mu_0}a_1^{\mu_1}a_2^{\mu_2} \dots b_0^{\nu_0}b_1^{\nu_1}b_2^{\nu_2} \dots ,$$

where

$$\mu_0 + \mu_1 + \mu_2 + \dots = n$$
$$\nu_0 + \nu_1 + \nu_2 + \dots = m.$$

These terms may be written in the form

$$k \cdot a_0^n b_0^m \cdot \left(\frac{a_1}{a_0}\right)^{\mu_1}\left(\frac{a_2}{a_0}\right)^{\mu_2} \dots \left(\frac{b_1}{b_0}\right)^{\nu_1}\left(\frac{b_2}{b_0}\right)^{\nu_2} \dots ,$$

hence the determinant in the form

$$R_d = a_0^n b_0^m \cdot \varphi(\alpha,\beta),$$

where $\varphi$ is an integral function of the $\alpha_i$ and $\beta_i$.

As the determinant vanishes for any common root

$$\alpha_i = \beta_k,$$

the function $\varphi$ is divisible by any $\alpha_i - \beta_k$, hence the determinant divisible by

$$a_0{}^n b_0{}^m \prod_{i,k} (\alpha_i - \beta_k). \quad [i = 1, \ldots, m; k = 1, \ldots, n$$

To find the quotient, we set

$$\begin{aligned} a_0{}^n b_0{}^m \prod_{i,k} (\alpha_i - \beta_k) &= a_0{}^n \prod_i [b_0(\alpha_i - \beta_1)(\alpha_i - \beta_2) \ldots (\alpha_i - \beta_n)] \\ &= a_0{}^n \prod_i g(\alpha_i) \qquad [i = 1, \ldots, m \\ &= (-1)^{mn} b_0{}^m \prod_k [a_0(\beta_k - \alpha_1) \ldots (\beta_k - \alpha_m)] \\ &= (-1)^{mn} b_0{}^m \prod_k f(\beta_k). \qquad [k = 1, \ldots, n \end{aligned}$$

Since $\prod_i g(\alpha_i)$ is homogeneous of total degree $m$ in the $b_i$ and $\prod_k f(\beta_k)$ is homogeneous of total degree $n$ in the $a_i$, it follows that $a_0{}^n b_0{}^m \prod_{i,k} (\alpha_i - \beta_k)$ is homogeneous of total degree $n$ in the $a_i$ and of total degree $m$ in the $b_i$. But so is the determinant, and their quotient can be only numerical. Comparing the leading term of the determinant, which is $a_0{}^n b_n{}^m$, with the corresponding term of $a_0{}^n b_0{}^m \prod_{i,k} (\alpha_i - \beta_k)$, which is

$$\begin{aligned} &a_0{}^n [b_0(-\beta_1)(-\beta_2) \ldots (-\beta_n)]^m \\ &= a_0{}^n [b_0(-1)^n \beta_1 \beta_2 \ldots \beta_n]^m \\ &= a_0{}^n \left[ b_0 \frac{b_n}{b_0} \right]^m \\ &= a_0{}^n b_n{}^m, \end{aligned}$$

we find that the quotient is 1, so that

$$R_d = a_0{}^n b_0{}^m \prod_{i,k} (\alpha_i - \beta_k) = R(f,g)$$

and the determinant appears to be the resultant itself expressed as function of the coefficients $a_i$ and $b_i$.

The resultant, expressed as function of the roots $\alpha_i$ and $\beta_i$, evidently is homogeneous of total degree $mn$ in these roots. Considering that $a_k$ is of degree $k$ in the $\alpha_i$ and $b_k$ is of degree $k$ in the $\beta_i$, we notice that the terms of the resultant expressed as function of the coefficients $a_i$ and $b_i$ are of constant weight in these coefficients, for

$$\begin{aligned} &\mu_1 + 2\mu_2 + 3\mu_3 + \dots \\ &+ \nu_1 + 2\nu_2 + 3\nu_3 + \dots = mn. \end{aligned}$$

Hence we have the proposition:

(37) **The resultant of two functions is homogeneous of total degree $mn$ in their roots and expressible as determinant of constant weight $mn$ in their coefficients, if $m$ and $n$ are the degrees of the functions.**

## §27. DISCRIMINANT

If $f'(x)$ denotes the derivative of

$$f(x) = a_0x^m + a_1x^{m-1} + \dots + a_m,$$

$R(f,f')$ is divisible by $a_0$ as we see from its determinant form which permits $a_0$ to be taken out of the first column. We set

$$\frac{1}{a_0} R(f,f') = l \,.\, D(f),$$

introducing the coefficient $l$ for an adjustment to be made presently, and call $D$ the **discriminant** of the function $f(x)$.

Since

$$R(f,f') = a_0^{m-1}f'(\alpha_1)f'(\alpha_2) \dots f'(\alpha_m),$$

we have

$$l \,.\, D(f) = a_0^{m-2}\prod_i f'(\alpha_i). \qquad [i = 1, \dots, m$$

But the derivative of $f(x)$ for $\alpha_i$ is

$$f'(\alpha_i) = a_0[(\alpha_i - \alpha_1)(\alpha_i - \alpha_2) \dots (\alpha_i - \alpha_{i-1})(\alpha_i - \alpha_{i+1}) \dots (\alpha_i - \alpha_m)].$$

This gives

$$\prod_i f'(\alpha_i) = (-1)^{\frac{m(m-1)}{2}} a_0^m \prod_{i \lessgtr k} (\alpha_i - \alpha_k)^2, \; [i, k = 1, \dots, m$$

with a coefficient taking care of the sign because there are $m(m-1)$ factors $\alpha_i - \alpha_k$ and every such factor appears twice with opposite sign in the left member while appearing twice with the same sign in the right member.

Substituting into the expression for $D$, we obtain

$$l \, . \, D(f) = (-1)^{\frac{m(m-1)}{2}} a_0^{2m-2} \prod_{i \lessgtr k} (\alpha_i - \alpha_k)^2. \quad [i, k = 1, \ldots, m$$

Now we set

$$l = (-1)^{\frac{m(m-1)}{2}},$$

and our formulas become definitively

$$\frac{1}{a_0} R(f,f') = (-1)^{\frac{m(m-1)}{2}} D(f),$$

$$D(f) = a_0^{2m-2} \prod_{i \lessgtr k} (\alpha_i - \alpha_k)^2.$$

Since

$$R(f,f') = 0$$

means a common root of $f(x)$ and $f'(x)$, and by proposition (12) such a common root is a multiple root of $f(x)$, we conclude that

(38) **the discriminant**

$$\boxed{D(f) = (-1)^{\frac{m(m-1)}{2}} \frac{R(f,f')}{a_0}}$$

**of a function $f(x)$ of degree $m$ vanishes whenever that function has a multiple root.**

For $a_0 = 1$ we obtain the non-homogeneous form of the discriminant, which we denote by $\Delta$:

$$\Delta(f) = \prod_{i \lessgtr k} (\alpha_i - \alpha_k)^2.$$

In case of the quadratic equation

$$f(x) = a_0 x^2 + a_1 x + a_2 = 0,$$

the discriminant is

$$D = a_1^2 - 4a_0 a_2.$$

In case of the cubic equation

$$f(x) = a_0x^3 + a_1x^2 + a_2x + a_3 = 0,$$

the discriminant is

$$D = a_1^2a_2^2 + 18a_0a_1a_2a_3 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2.$$

We may notice that already

$$\sqrt{\Delta} = \prod_{i \lessgtr k} (\alpha_i - \alpha_k)$$

disappears when two roots are alike; but it is not symmetric in the $\alpha_i$ and hence not rationally expressible in terms of the $a_i$.

We may also notice that it makes a difference with $\sqrt{\Delta}$ whether we set $i < k$ or $i > k$, while it makes no difference with $\Delta$, for

$$\prod_{i<k} (\alpha_i - \alpha_k) = (-1)^{\frac{m(m-1)}{2}} \prod_{i>k} (\alpha_i - \alpha_k)$$

where

$$\begin{aligned}\prod_{i<k} (\alpha_i - \alpha_k) = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \ . \ . \ . \ (\alpha_1 - \alpha_m)\\ (\alpha_2 - \alpha_3) \ . \ . \ . \ (\alpha_2 - \alpha_m)\\ . \ . \ . \ . \ . \ . \ .\end{aligned}$$

and

$$\begin{aligned}\prod_{i>k} (\alpha_i - \alpha_k) = (\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1) \ . \ . \ . \ (\alpha_m - \alpha_1)\\ (\alpha_3 - \alpha_2) \ . \ . \ . \ (\alpha_m - \alpha_2)\\ . \ . \ . \ . \ . \ . \ .\end{aligned}$$

# CHAPTER VI

## COMPOSITION OF SYMMETRIC GROUP

### §28. COMPOSITION-SERIES

A complete series of normal subgroups, each the greatest such group in the preceding, is called the **composition-series** of a group. The composition-series of the symmetric group begins with the symmetric group itself and ends with identity as every composition-series necessarily does.

The indices of the composition-series, each one taken as the index of a group in the group preceding it, are called the **factors of composition.**

To find a composition-series, it is useful to know that normal subgroups of a group $G$ are composed of all permutations which are common to conjugate subgroups of $G$.

For suppose such conjugate subgroups of $G$ are

$$H_1, H_2, \ldots, H_j,$$

The permutations which are common to these subgroups form a group, since with any two of them contained in $H_i$ also their product is contained in $H_i$ by the very definition of a group. And since the transforms of $H_1$ by permutations in $G$ are the same conjugate subgroups

$$H_1, t_2^{-1}H_1t_2, \ldots, t_j^{-1}H_1t_j,$$

the group of common permutations, which is part of any one of them as it is of $H_1$, is identical with its transforms and hence a normal subgroup of $G$.

A group of all permutations which are common to conjugate subgroups is called their **greatest common subgroup**[1] and denoted by $D$, so that we have the proposition:

(39) **a greatest common subgroup is normal in the group.**[2]

The question may occur why there cannot be distinct conjugates $D'$ and $D''$ such that $D'$ of $H_1$ is transformed into $D''$ of

[1] Called "Durchschnitt" in German.

[2] Compare §54.

$H_i$ while $D'$ of $H_i$ comes from $D''$ of $H_1$. But this implies the presence of both $D'$ and $D''$ in any $H_i$ and more common permutations than we have in fact.

Thus the conjugate subgroups in the example of §19 have the greatest common subgroup

$$D = 1,\ (12)(34),\ (13)(24),\ (14)(23)$$

which is normal in $S$, while

$$\begin{aligned} H_1 &= 1,\ (12)(34) \\ H_2 &= 1,\ (13)(24) \\ H_3 &= 1,\ (14)(23) \end{aligned}$$

are common subgroups but not normal since they are not greatest.

## §29. ALTERNATING FUNCTION

The greatest possible subgroup of the symmetric group $S$ on $n$ letters $x_i$ is one of index

$$j = 2$$

and order

$$r = \frac{n!}{2}.$$

It exists by proposition (20) because there exist two-valued functions of these $x_\iota$, which are called **alternating functions.**[1]

Suppose that $A_i$ is an alternating function with no other values than $A_1$ and $A_2$. If the group of $A_1$ is

$$\{A\} = 1, s_2, \ . \ . \ . \ , s_r$$

and the permutation $t$ is not in it, then the permutations of

$$\{A\}t = t, s_2t, \ . \ . \ . \ , s_rt$$

convert $A_1$ into $A_2$ and

$$\{S\} = \{A\} + \{A\}t.$$

Since both sets $At$ and $tA$ contain all permutations that are in $S$ and not in $A$, we have

$$\{A\}t = t\{A\}$$

and

$$t^{-1}\{A\}t = \{A\}.$$

[1] Cf. §8.

Hence $A$ is normal: it is the group of $A_2$ as it is the group of $A_1$, while the permutations of $At$ convert $A_2$ into $A_1$ as they convert $A_1$ into $A_2$.

As a permutation of $S$ either leaves the functions $A_1$ and $A_2$ unaltered or interchanges them, the sum

$$A_1 + A_2$$

is a symmetric function while the difference

$$A_1 - A_2 = \Phi$$

has the conjugate value

$$A_2 - A_1 = -\Phi$$

and is an alternating function.

Taking now a permutation that alters $\Phi$, we resolve it into transpositions. As we successively apply these transpositions to $\Phi$, we must strike one that changes the sign of $\Phi$, for the permutation does so. If it is $(x_1x_2)$, so that

$$\Phi(x_1, x_2, \ldots, x_n) = -\Phi(x_2, x_1, \ldots, x_n),$$

we set

$$x_1 = x_2;$$

this gives

$$\Phi(x_1, \ldots, x_n) = -\Phi(x_1, \ldots, x_n) = 0,$$

since both members differ by their sign alone.

It appears that $\Phi$ is divisible by $x_1 - x_2$. But then $\Phi^2$ is divisible by $(x_1 - x_2)^2$ and, as it evidently is symmetric, by every $(x_i - x_k)^2$. Hence $\Phi$ in turn is divisible by every $x_i - x_k$ and equals

$$\sqrt{\Delta} = \prod_{i \lessgtr k} (x_i - x_k)$$

itself or multiplied by a symmetric function. Setting

$$A_1 + A_2 = 2S_1$$
$$\Phi = A_1 - A_2 = 2S_2\sqrt{\Delta},$$

we find that

(40) **the general form of alternating functions i**

$$\boxed{\begin{aligned} A_1 &= S_1 + S_2\sqrt{\Delta} \\ A_2 &= S_1 - S_2\sqrt{\Delta} \end{aligned}}.$$

With the alternating functions exists the group of these functions which is called the **alternating group** and is denoted by $A$ or, to guard against confusion, by $\{A\}$:

(41) **The alternating group is a normal subgroup of index two in the symmetric group.**

It is the greatest subgroup of the symmetric group, containing one half of its permutations. The simplest function that belongs to it is the root $\sqrt{\Delta}$ of the discriminant.

## §30. ALTERNATING GROUP

Every transposition alters the sign of $\sqrt{\Delta}$ and the value of an alternating function, as we can verify on any example. Writing subscripts only, we have in five letters $x_i$, for instance:

$$\begin{aligned}\sqrt{\Delta} = (1-2)(1-3)(1-4)&(1-5)\\ (2-3)(2-4)&(2-5)\\ (3-4)&(3-5)\\ &(4-5).\end{aligned}$$

By the transposition (24), say, the factors $(1-2)$ and $(2-5)$ only change place with $(1-4)$ and $(4-5)$, the factor $(2-3)$ changes place with $(3-4)$ and also sign. While this leaves $\sqrt{\Delta}$ unaltered, there is one factor, the factor $(2-4)$ containing the numbers of the transposition, which changes its own sign and that of $\sqrt{\Delta}$.

Hence we conclude that any even number of transpositions leaves $\sqrt{\Delta}$ unaltered while any odd number of them alters its sign. If then some permutation once breaks up into an even number of transpositions, it always does so since it cannot alter and not alter the sign of $\sqrt{\Delta}$. Such a permutation is called an **even permutation**; while another which is formed by an odd number of transpositions is called **odd.**

(42) **A circular permutation, or the cycle of a non-circular permutation, is odd or even according as its degree is even or odd; a non-circular permutation is odd or even according as it contains an odd or even number of odd cycles.**

For instance:

$$(123) = (12)(13)$$

is even while

$$(1234) = (12)(13)(14)$$

is odd, and

$$(1234)(56)$$

is even because it contains two odd cycles. Two similar permutations evidently are both odd or both even.

It is clear that

(43) **the alternating group is composed of all even permutations in the symmetric group,**

all those that leave unaltered the sign of $\sqrt{\Delta}$. We readily admit that such permutations form a group, for the product of even permutations is again an even permutation; also that this group is normal, for the transform of an even permutation is by proposition (30) again even.

Every permutation on $n$ letters $x_i$ can by proposition (33) be represented as the product of transpositions which are in the set $(x_1x_i)_2^n$; every even permutation therefore as the product of two such transpositions, which is to say as the product of circular permutations

$$(x_1x_ix_k) = (x_1x_i)(x_1x_k)$$

of order three. But we can easily verify that

$$(x_1x_ix_k) = (x_1x_2x_i)^2(x_1x_2x_k),$$

whence the permutations of the alternating group are resolvable into cycles of order three and, if we so choose, into such as give the $n - 2$ permutations

$$(x_1x_2x_i)_3^n = (x_1x_2x_3),\ (x_1x_2x_4),\ \ldots,\ (x_1x_2x_n).$$

It follows that

(44) **the alternating group on $n$ letters $x_i$ is generated by the $n - 2$ independent permutations $(x_1x_2x_i)_3^n$:**

$$\boxed{A^n = \{(x_1x_2x_i)_3^n\}};$$

and a group including these permutations is the alternating if not the symmetric group.

For the lowest degrees the alternating groups are:

$$A^2 = 1$$
$$A^3 = 1,\ (123),\ (132)$$

$A^4 =$

| 1 | (12)(34) | (13)(24) | (14)(23) |
|---|---|---|---|
| (123)<br>(132) | (243)<br>(143) | (142)<br>(234) | (134)<br>(124) |

.

If we denote the numbers

$$1, 2, 3, 4, \ldots$$

in any order whatever by

$$i_1, i_2, i_3, i_4, \ldots$$

we can assert that the two permutations

$$t_1 = \begin{pmatrix} 1\,2\,3\,4 & \ldots \\ i_1 i_2 i_3 i_4 & \ldots \end{pmatrix}$$

and

$$t_2 = \begin{pmatrix} 1\,2\,3\,4 & \ldots \\ i_2 i_1 i_3 i_4 & \ldots \end{pmatrix}$$

cannot both be odd or even, for their product $t_1t_2$ is odd since the product

$$t_1t_2^{-1} = (12)$$

is odd and the permutations $t_2$ and $t_2^{-1}$ are similar by proposition (18).

Hence either $t_1$ or $t_2$ is in the alternating group, and

(45) **a normal subgroup of the alternating group contains with one circular permutation of order three every possible such permutation.**

For suppose it contains the permutation

$$s = (123).$$

If the permutation $t_1$ is in the alternating group, its normal subgroup contains the permutation

$$t_1^{-1}st_1 = (i_1i_2i_3);$$

and if the permutation $t_2$ is in the alternating group, its normal subgroup contains the permutation

$$t_2^{-1}s^2t_2 = (i_1i_2i_3).$$

Thus it cannot help containing any circular permutation $(i_1i_2i_3)$ of order three.

## §31. COMPOSITION OF $S$ AND $A$

So far we know about the composition-series of the symmetric group on $n$ letters $x_i$ that it is

$$S \quad A \ . \ . \ . \ 1.$$

We now ask the question that could make our hearts beat quicker if we knew its significance: Are there in the symmetric group normal subgroups other than the alternating group? And are there normal subgroups in the alternating group?

Supposing there is in either of these groups a normal subgroup $N$, so that we have

$$t_i^{-1}Nt_i = N$$

at least for every permutation $t_i$ of the alternating group, we search for those permutations of $N$ that have the lowest degree. Of course, we shall disregard identity and not count toward the degree letters that remain fixed.

The permutations of lowest degree cannot be circular of degree more than three, nor non-circular containing a cycle of degree more than three, for a permutation

$$s_1 = (1234 \ . \ . \ . \ ) \ . \ . \ .$$

of $N$ is transformed by the permutation

$$t = (123)$$

of $A$ into the permutation

$$t^{-1}s_1t = (2314 \ . \ . \ . \ ) \ . \ . \ . = s_2$$

of $N$. Hence $N$ contains also the permutation

$$s_1s_2^{-1} = (2)(31 \ . \ . \ . \ ) \ . \ . \ .$$

which is of lower degree than $s_1$ since the letter with subscript 2 is excluded and no letter added.

The permutations of lowest degree cannot be non-circular with a cycle of degree three, for a permutation

$$s_1 = (123)(4 \ . \ . \ . \ ) \ . \ . \ .$$

of $N$ is transformed by the permutation

$$t = (234)$$

of $A$ into the permutation

$$t^{-1}s_1t = (134)(2 \ . \ . \ . \ ) \ . \ . \ . \ = s_2$$

of $N$. Hence $N$ contains also the permutation

$$s_1s_2 = (3)(24 \ . \ . \ . \ ) \ . \ . \ .$$

which is of lower degree than $s_1$.

Circular permutations of degree three qualify as permutations of lowest degree in $N$. But containing one of them, $N$ contains them all by proposition (45) and is by proposition (44) the alternating if not the symmetric group.

Likewise single transpositions qualify; but containing such, $N$ is a subgroup of the symmetric and not the alternating group, contains them all by proposition (31) and is by proposition (33) the symmetric group itself.

Non-circular permutations of degree more than four with cycles of degree two are impossible, for a permutation

$$s_1 = (12)(34) \ . \ . \ .$$

of $N$ is transformed by the permutation

$$t = (345)$$

of $A$ into the permutation

$$t^{-1}s_1t = (12)(45) \ . \ . \ . \ = s_2$$

of $N$. Hence $N$ contains also the permutation

$$s_1s_2 = (1)(2)(35 \ . \ . \ . \ ) \ . \ . \ .$$

which lost two letters although it may have gained one and is of lower degree as compared with $s_1$.

This conclusion is not valid, however, for a group of degree four. The two possible transforms of

$$s_1 = (12)(34)$$

then are

$$s_2 = (13)(24)$$
$$s_3 = (14)(23),$$

and these permutations together with identity constitute the **quadratic group**[1]

$$V = 1, (12)(34), (13)(24), (14)(23)$$

of degree and order four which is normal in both the symmetric and alternating groups of degree four.

The disappointing result of our investigation is this:

(46) **The symmetric group has no normal subgroup other than the alternating group, and the alternating group has no normal subgroup at all. Only the symmetric and alternating groups of degree four are exceptions.**

Hence the symmetric group is always composite; but the alternating group is always simple, except when its degree is four.

The quadratic group is unique in many ways; although not circular, it is composed of commutative permutations: for instance,

$$(12)(34)\,.\,(13)(24) = (13)(24)\,.\,(12)(34) = (14)(23).$$

As a six-valued function belonging to $V$ we mention

$$\psi = (x_1 - x_2)(x_3 - x_4);$$

another example occurs in §39. But not every six-valued function belongs to $V$; thus the function

$$\xi = x_1 + x_2 - x_3 - x_4$$

belongs to the croup

$$W = 1, (12), (34), (12)(34).$$

## §32. SUBGROUPS OF $S$ AND $A$

If the symmetric group has no other normal subgroup than the alternating group, it has other subgroups that are not normal. But

(47) **the symmetric group on $n$ letters $x_i$ has no subgroup of index between 2 and $n$, except when $n = 4$.**

This implies that a function in $n$ letters $x_i$ which has fewer than $n$ values cannot have more than 2 values, unless in the unruly case of $n = 4$.

[1] Called "Vierergruppe" in German.

For suppose that a function $\psi_1$ takes under the symmetric group $S$ fewer than $n$ conjugate values, and suppose that these values are

$$\psi_1, \psi_2, \ldots, \psi_j$$

belonging to conjugate subgroups $[j < n$

$$H_1, H_2, \ldots, H_j$$

of $S$. The $n!$ permutations on the $x_i$ which are in $S$ can do no more than interchange the $\psi_i$, as it was explained in §16, and interchange them in $j!$ possible ways only.

Since $j!$ is less than $n!$, there must be distinct permutations on the $x_i$ that operate the same permutation on the $\psi_i$. Let $t$ and $u$ be two of them, then

$$s = tu^{-1}$$

is a permutation different from identity that leaves the $\psi_i$ unaltered. But permutations which do so are contained in every $H_i$; they are the only common permutations of the $H_i$ and compose by proposition (39) a normal subgroup

$$D = 1, s_2, \ldots, s_r$$

of the symmetric group $S$.

By proposition (46), this normal subgroup $D$ can be the alternating group alone, for it is not identity. Containing then the alternating group, the subgroup $H_1$ is identical with it if not symmetric; and belonging to $H_1$, the function $\psi_1$ is an alternating if not a symmetric function. Having less than $n$ values, it cannot have more than two values, which proves the proposition.

Likewise, the number $n!/2$ of permutations in the alternating group $A$ is greater than $j!$ for $j$ less than $n$,[1] if a function $\psi_1$ takes $j$ values $\psi_i$ under $A$, and there must be distinct permutations in $A$ which operate the same permutation on the $\psi_i$.

Since a subgroup of index between 1 and $n$ in the alternating group then calls for a normal subgroup between the alternating group and identity, and such a subgroup by proposition (46) does not exist, it appears that

(48) **the alternating group on $n$ letters $x_i$ has no subgroup of index between 1 and $n$, except when $n = 4$.**

[1] This is meant for $n > 2$.

The subgroup of index $n$ in the symmetric group is therefore not a subgroup of the alternating group, as we shall observe presently when identifying such a subgroup.

Another way of putting the proof of our propositions follows. The number $\rho$ of permutations between the $\psi_i$ under $S$ is

$$\rho = n!/r,$$

while necessarily

$$j! \geqq \rho.$$

As $j > 2$ means[1]

$$r = 1,$$

we have

$$j \geqq n.$$

An $n$-valued function $\psi_1$ in $n$ letters $x_i$ is readily constructed if we set $\psi_1$ equal to $x_1$ or to the sum of all $x_i$ except $x_1$. Thus we have for $n = 3$ the three-valued functions

$$\psi_1 = x_1, \qquad \psi_2 = x_2, \qquad \psi_3 = x_3,$$

or

$$\psi_1 = x_2 + x_3, \quad \psi_2 = x_1 + x_3, \quad \psi_3 = x_1 + x_2.$$

This verifies that a subgroup of index $n$ in $S^n$ is $S^{n-1}$, the group for instance of $\psi_1$ containing all permutations which act upon the $x_i$ other than $x_1$.

In the exceptional case of four letters $x_i$ there is between the alternating group and identity a normal subgroup of order

$$r = 4$$

in the symmetric group. Hence we have

$$j \geqq 3$$

and look for a function with

$$2 < j = 3 < 4.$$

One such function is

$$\psi_1 = x_1x_2 + x_3x_4,$$

which occurred in §17.

The number $\rho$ of permutations between the $\psi_i$ under $A$ is

$$\rho = \frac{n!}{2}/r,$$

while again

$$j! \geqq \rho.$$

As $j > 1$ means[1]

$$r = 1,$$

[1] By proposition (46).

we have

$$j! \geqq \frac{n!}{2}.$$

The exceptional case

$$r = 4$$

when $n = 4$ gives

$$j! \geqq 6$$

and permits

$$1 < j = 3 < 4.$$

The same function

$$\psi_1 = x_1x_2 + x_3x_4$$

will serve as an example.

We note that our results may also be expressed by saying that symmetric and alternating functions exist for any number of letters; three-valued functions exist in three and four letters only; and after that no $n$-valued function can be constructed in more than $n$ letters.

## §33. GROUP ON FUNCTIONS

We noticed that under the partitions of a group

$$G = N + Nt_2 + \ . \ . \ . \ + Nt_j$$

with respect to a normal subgroup $N$ conjugate values of a function interchange in as many ways as there are partitions if they belong to $N$ or to groups[1] containing $N$ as greatest common subgroup.

We have to add that the permutations between such conjugate values of a function, as effected by the permutations of $G$, form a group, which we express in the proposition:

(49) **The permutations under $G$ between conjugate functions $\psi_i$ belonging to a normal subgroup $N$ of $G$ or to groups containing $N$ as greatest common subgroup compose a group $\Gamma$.**

For suppose that these permutations are

$$1, \tau_2, \ . \ . \ . \ , \tau_j$$

interchanging the conjugate values

$$\psi_1, \psi_2, \ . \ . \ . \ , \psi_j$$

[1] Whether subgroups of $G$ or not.

as permutations in the partitions

$$N, Nt_2, \ . \ . \ . \ , Nt_j$$

of $G$ do. Set

$$\tau_a = \begin{pmatrix} \psi_1 \psi_2 & . \ . \ . & \psi_j \\ \psi_{1a}\psi_{2a} & . \ . \ . & \psi_{ja} \end{pmatrix} = \begin{pmatrix} \psi_i \\ \psi_{ia} \end{pmatrix}$$

and

$$\tau_b = \begin{pmatrix} \psi_i \\ \psi_{ib} \end{pmatrix} = \begin{pmatrix} \psi_{ia} \\ \psi_{iab} \end{pmatrix},$$

which we can do because the $\psi_{ia}$ represent in some order or other all the $\psi_i$. Then we have

$$\tau_a \tau_b = \begin{pmatrix} \psi_i \\ \psi_{iab} \end{pmatrix};$$

and since the permutation

$$t_a t_b = t_c$$

is contained in $G$, the permutation

$$\tau_a \tau_b = \tau_c$$

is contained among the $\tau_i$ which consequently compose a group

$$\Gamma = 1, \tau_2, \ . \ . \ . \ , \tau_j.$$

This we illustrate on the permutations between the functions

$$\psi_1 = x_1x_2 + x_3x_4$$
$$\psi_2 = x_1x_3 + x_2x_4$$
$$\psi_3 = x_1x_4 + x_2x_3$$

conjugate under the symmetric group on the $x_i$:

| $\Gamma$ on the $\psi_i$ | | $S$ on the $x_i$ | |
|---|---|---|---|
| 1 | $\psi_1\psi_2\psi_3$ | 1, (12)(34), (13)(24), (14)(23) $= N$ | |
| $\tau_2 = (123)$ | $\psi_2\psi_3\psi_1$ | (234), (132), (143), (124) $= Nt_2$ | $t_2 = (234)$ |
| $\tau_3 = (132)$ | $\psi_3\psi_1\psi_2$ | (243), (142), (123), (134) $= Nt_3$ | $t_3 = (243)$ |
| $\tau_4 = (12)$ | $\psi_2\psi_1\psi_3$ | (23), (1342), (1243), (14) $= Nt_4$ | $t_4 = (23)$ |
| $\tau_5 = (13)$ | $\psi_3\psi_2\psi_1$ | (24), (1432), (13), (1234) $= Nt_5$ | $t_5 = (24)$ |
| $\tau_6 = (23)$ | $\psi_1\psi_3\psi_2$ | (34), (12), (1423), (1324) $= Nt_6$ | $t_6 = (34)$ |

# CHAPTER VII

# THEORY OF LAGRANGE

## §34. RESOLVENT EQUATION

In the endeavor to find a solution of the general equation

$$f(x) = x^n - c_1x^{n-1} + \dots \pm c = 0$$

we now turn to functions of its roots

$$x_1, x_2, \dots, x_n,$$

for we know from chapter two that the computation of such functions may imply the solution of the general equation.

Any such function takes a certain number of conjugate values under every group containing the group of the function. An equation whose roots such conjugate values of a function are is called a **resolvent equation or resolvent** of the general equation, and the following proposition applies:

(50) **The conjugate values which a function takes under a group to whose subgroup the function belongs are roots of a resolvent equation whose degree equals the index of the subgroup in the group and whose coefficients belong to the group.**

For suppose that some function $\psi_1$ of the $x_i$ belongs to a subgroup $H$ of index $j$ in the group $G$ on the $x_i$. Its conjugate values

$$\psi_1, \psi_2, \dots, \psi_j$$

under $G$ are roots of the resolvent equation

$$r(\psi) = (\psi - \psi_1)(\psi - \psi_2) \dots (\psi - \psi_j) = 0$$

of degree $j$ with coefficients

$$\psi_1 + \psi_2 + \psi_3 + \dots + \psi_j$$
$$\psi_1\psi_2 + \psi_1\psi_3 + \dots + \psi_{j-1}\psi_j$$
$$\dots\dots\dots$$
$$\psi_1\psi_2\psi_3 \dots \psi_j$$

which are symmetric in the $\psi_i$. Since a permutation of $G$ can do no more than interchange the $\psi_i$, as explained in §16, it leaves the coefficients of the resolvent equation unaltered. This no permutation outside $G$ does, and the proposition follows.

In this connection we say that $r(\psi) = 0$ is a resolvent equation for $H$ and that $G$ is the group of this resolvent equation.

## §35. LAGRANGE'S THEOREM

That the solution of a resolvent equation may imply the solution of the general equation follows from a proposition known as **Lagrange's Theorem**[1] and published in a celebrated contribution of Lagrange to the Memoirs of the Academy of Berlin for 1770–71:

(51) **If a rational function $\varphi_1$ in the roots $x_i$ of the general equation remains unaltered by all those permutations on the $x_i$ that leave another rational function $\psi_1$ of the $x_i$ unaltered, then the function $\varphi_1$ is rationally expressible in terms of the function $\psi_1$ and the coefficients of the general equation.**

This is to say that the function $\varphi_1$ can be computed from the function $\psi_1$ and the coefficients of the general equation by rational operations. We note incidentally that by the conditions of the theorem the function $\psi_1$ belongs to the group of the function $\varphi_1$ or to one of its subgroups.

Let $\varphi_1$ and $\psi_1$ be two rational functions of the $x_i$ belonging first to the same group $H$ of index $j$ in the symmetric group $S$ on the $x_i$. We shall prove that $\varphi_1$ is rationally expressible in terms of $\psi_1$ and the coefficients $c_i$ of the general equation.

The conjugate values

$$\varphi_1, \varphi_2, \ . \ . \ . \ , \varphi_j$$
$$\psi_1, \psi_2, \ . \ . \ . \ , \psi_j$$

of $\varphi_1$ and $\psi_1$ under $S$ can be arranged so that permutations of $S$ converting $\varphi_i$ into $\varphi_k$ also convert $\psi_i$ into $\psi_k$. The function

$$\psi_1{}^k\varphi_1 + \psi_2{}^k\varphi_2 + \ . \ . \ . \ + \psi_j{}^k\varphi_j$$

[1] Compare the statement of Lagrange's Theorem in §71. Another proof is given in §69. Lagrange lived 1736–1813.

then is symmetric in the $x_i$ since no permutation on the $x_i$ can do more than interchange its terms. Consequently it is rational in the $c_i$ by proposition (35), and we have a system

$$\begin{aligned}
\varphi_1 + \varphi_2 + \cdots + \varphi_j &= r_0(c_i) \\
\psi_1\varphi_1 + \psi_2\varphi_2 + \cdots + \psi_j\varphi_j &= r_1(c_i) \\
\psi_1^2\varphi_1 + \psi_2^2\varphi_2 + \cdots + \psi_j^2\varphi_j &= r_2(c_i) \\
\cdots\cdots\cdots \\
\psi_1^{j-1}\varphi_1 + \psi_2^{j-1}\varphi_2 + \cdots + \psi_j^{j-1}\varphi_j &= r_{j-1}(c_i)
\end{aligned}$$

of $j$ equations linear in the $\varphi_i$. We need no more equations to solve for $\varphi_1$; and we can have no more since powers of $\psi_i$ higher than $j - 1$ can be eliminated by rational operations. For $\psi_i$ satisfies by proposition (50) an equation

$$r(\psi) = \psi^j + A_1\psi^{j-1} + A_2\psi^{j-2} + \cdots = 0$$

with coefficients that are symmetric in the $x_i$ and as such rational in the $c_i$, whence

$$\begin{aligned}
\psi_i^j &= -A_1\psi_i^{j-1} - A_2\psi_i^{j-2} - \cdots \\
\psi_i^{j+1} &= -A_1\psi_i^j - A_2\psi_i^{j-1} - \cdots \\
&= (A_1^2 - A_2)\psi_i^{j-1} + (A_1A_2 - A_3)\psi_i^{j-2} + \cdots \\
\cdots\cdots\cdots
\end{aligned}$$

Solving the system of equations in the $\varphi_i$ given above for $\varphi_1$, we obtain

$$\varphi_1 = \frac{\begin{vmatrix} r_0 & 1 & 1 & \cdots & 1 \\ r_1 & \psi_2 & \psi_3 & \cdots & \psi_j \\ r_2 & \psi_2^2 & \psi_3^2 & \cdots & \psi_j^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{j-1} & \psi_2^{j-1} & \psi_3^{j-1} & \cdots & \psi_j^{j-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \psi_1 & \psi_2 & \psi_3 & \cdots & \psi_j \\ \psi_1^2 & \psi_2^2 & \psi_3^2 & \cdots & \psi_j^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_1^{j-1} & \psi_2^{j-1} & \psi_3^{j-1} & \cdots & \psi_j^{j-1} \end{vmatrix}}$$

Since the determinant of the denominator vanishes for every

$$\psi_i = \psi_k, \qquad [i \neq k$$

this determinant is divisible by every $\psi_i - \psi_k$ and hence by

$$\prod(\psi_i - \psi_k) = \sqrt{\Delta_\psi},$$

where we take

$$i > k.$$

Having $j$ values $\psi_i$, we can pick $\psi_i - \psi_k$ such that $i > k$ in $j(j-1)/2$ different ways, whence $\prod(\psi_i - \psi_k)$ is homogeneous of total degree $j(j-1)/2$ in the $\psi_i$. But so is the determinant as

$$1 + 2 + \ldots + (j-1) = \frac{j(j-1)}{2},$$

and their quotient can be only numerical. From the leading term of the determinant, as compared with the corresponding term of $\prod(\psi_i - \psi_k)$, it is seen to be 1, whence

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ \psi_1 & \psi_2 & \psi_3 & \ldots & \psi_j \\ \psi_1^2 & \psi_2^2 & \psi_3^2 & \ldots & \psi_j^2 \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \psi_1^{j-1} & \psi_2^{j-1} & \psi_3^{j-1} & \ldots & \psi_j^{j-1} \end{vmatrix} = \sqrt{\Delta_\psi}$$

Denoting now by $T$ the determinant of the numerator, we may set

$$\varphi_1 = \frac{T}{\sqrt{\Delta_\psi}} = \frac{T \cdot \sqrt{\Delta_\psi}}{\Delta_\psi},$$

where $\Delta_\psi$ is symmetric in the $\psi_i$ and hence the $x_i$ and as such rational in the $c_i$. To complete our proof, it remains to investigate the numerator

$$T \cdot \sqrt{\Delta_\psi} = \begin{vmatrix} r_0 & 1 & 1 & \ldots & 1 \\ r_1 & \psi_2 & \psi_3 & \ldots & \psi_j \\ r_2 & \psi_2^2 & \psi_3^2 & \ldots & \psi_j^2 \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ r_{j-1} & \psi_2^{j-1} & \psi_3^{j-1} & \ldots & \psi_j^{j-1} \end{vmatrix} \cdot \sqrt{\Delta_\psi}$$

It is a function symmetric in the conjugate values

$$\psi_2, \psi_3, \ldots, \psi_j$$

other than $\psi_1$, for interchanging any two such values $\psi_i$ we alter the sign of both $T$ and $\sqrt{\Delta_\psi}$ but do not alter their product. To examine such a function, we take the equation

$$(\psi - \psi_1)(\psi - \psi_2) \ldots (\psi - \psi_j)$$
$$= \psi^j + A_1\psi^{j-1} + A_2\psi^{j-2} + \ldots + A_j = 0$$

with coefficients rational in the $c_i$ and dividing out $\psi - \psi_1$ obtain the equation

$$\begin{aligned}(\psi - \psi_2) \ldots (\psi - \psi_j) \\ = \psi^{j-1} + (\psi_1 + A_1)\psi^{j-2} + (\psi_1^2 + A_1\psi_1 + A_2)\psi^{j-3} + \ldots \\ = \psi^{j-1} + B_1\psi^{j-2} + B_2\psi^{j-3} + \ldots = 0,\end{aligned}$$

where

$$\begin{aligned}B_1 &= \psi_1 + A_1 \\ B_2 &= \psi_1^2 + A_1\psi_1 + A_2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{aligned}$$

Hence it appears that a symmetric function of the roots $\psi_i$ other than $\psi_1$, rationally expressible in terms of the elementary symmetric functions $B_i$ of those roots, is expressible so in terms of $\psi_1$ and the $A_i$, and therefore expressible so in terms of $\psi_1$ and the $c_i$:

$$\text{Sym. } (\psi_2, \ldots, \psi_j) = \text{Rat. } (\psi_1, c_i).$$

Since the numerator in the expression for $\varphi_1$ is such a function and its coefficients are rational with the $r_i$, we may set

$$\varphi_1 = \frac{\text{Rat. } (\psi_1, c_i)}{\Delta_\psi}$$

or

$$\varphi_1 = R(\psi_1, c_i)$$

with $R$ integral in $\psi_1$.

This proves Lagrange's Theorem for functions belonging to the same group. Inasmuch as the coefficients $c_i$ of the general equation are to be regarded as rational, we may set also

$$\varphi_1 = R(\psi_1).$$

Since powers of $\psi_1$ higher than $j - 1$ can be eliminated, the expression for $\varphi_1$ can be reduced to the form

$$\varphi_1 = \frac{R_1\psi_1^{j-1} + R_2\psi_1^{j-2} + \ldots + R_j}{\Delta_\psi}.$$

### §36. LAGRANGE'S THEOREM, *Continued*

If the function $\psi_1$ belongs to $H$ as it did before, where $H$ is a subgroup of index $k$ in the group $G$, and the function $\varphi_1$ now belongs to $G$, we have as corresponding conjugate values of $\varphi_1$ and $\psi_1$ under $G$:

$$\begin{array}{llllllll}\varphi_1, & \varphi_2, & \ldots, & \varphi_{j/k}, & \varphi_1, & & \ldots, & \varphi_{j/k} \\ \psi_1, & \psi_2, & \ldots, & \psi_{j/k}, & \psi_{j/k+1}, & & \ldots, & \psi_j;\end{array}$$

and we have as equations in the $\varphi_i$:

$$\begin{array}{l} \varphi_1 + \varphi_2 + \ldots + \varphi_{j/k} + \varphi_1 + \ldots + \varphi_{j/k} = r_0 \\ \psi_1\varphi_1 + \psi_2\varphi_2 + \ldots + \psi_{j/k}\varphi_{j/k} + \psi_{j/k+1}\varphi_1 + \ldots + \psi_j\varphi_{j/k} = r_1 \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

The $\varphi_i$ are not all distinct, but this does not interfere with the computation of $\varphi_1$ which is to follow, and we have again

$$\varphi_1 = R(\psi_1, c_i)$$

or

$$\varphi_1 = R(\psi_1).$$

This completes the proof of Lagrange's Theorem.

In the last case

$$\psi_1 \neq R(\varphi_1, c_i),$$

for a computation of $\psi_1$ from equations constructed for the $\psi_i$ leads to determinants vanishing with like columns in the $\varphi_i$.

Conversely, if

$$\varphi_1 = R(\psi_1, c_i),$$

then a permutation leaving $\psi_1$ unaltered leaves unaltered also $\varphi_1$, and we infer that the group of $\varphi_1$ contains that of $\psi_1$. If moreover

$$\psi_1 = R(\varphi_1, c_i),$$

then also the group of $\psi_1$ contains that of $\varphi_1$, whence both groups are identical. Thus the Theorem of Lagrange presents a necessary and sufficient condition.[1]

If the function $\psi_1$ belongs to the subgroup $H$ of $G$, then $G$ can do no more than permute the conjugate values $\psi_i$ that $\psi_1$ takes under $G$. Hence symmetric functions of such values are unaltered by $G$; yet they may belong to $G$ and may not.

That they may not appears from an example. Conjugate values of a function belonging to identity are:

| | | | |
|---|---|---|---|
| $\psi_1 = 2x_1 - x_2$ | 1 | ↑ | ↑ |
| $\psi_2 = 2x_2 - x_3$ | (123) | $G$ | |
| $\psi_3 = 2x_3 - x_1$ | (132) | ↓ | $S$ |
| $\psi_4 = 2x_2 - x_1$ | (12) | | |
| $\psi_5 = 2x_3 - x_2$ | (13) | | |
| $\psi_6 = 2x_1 - x_3$ | (23) | | ↓ |

[1] The rest of this paragraph may be omitted on first reading.

These conjugate values are permuted so:

| | | | | |
|---|---|---|---|---|
| ↑ | 1 | $\psi_1\psi_2\psi_3\psi_4\psi_5\psi_6$ | 1 | ↑ |
| | (123)(456) | $\psi_2\psi_3\psi_1\psi_5\psi_6\psi_4$ | (123) | |
| Γ on the $\psi_i$ | (132)(465) | $\psi_3\psi_1\psi_2\psi_6\psi_4\psi_5$ | (132) | $S$ on the $x_i$ |
| | (14)(26)(35) | $\psi_4\psi_6\psi_5\psi_1\psi_3\psi_2$ | (12) | |
| | (15)(24)(36) | $\psi_5\psi_4\psi_6\psi_2\psi_1\psi_3$ | (13) | |
| ↓ | (16)(25)(34) | $\psi_6\psi_5\psi_4\psi_3\psi_2\psi_1$ | (23) | ↓ |

The symmetric functions

$$\psi_1 + \psi_2 + \psi_3$$

and

$$\psi_4 + \psi_5 + \psi_6$$

of three $\psi_i$ conjugate under $G$ are unaltered by $G$ and conjugate in $S$. Yet they belong not to $G$ but to $S$ because they are symmetric also in the $x_i$:[1]

$$\psi_1 + \psi_2 + \psi_3 = \psi_4 + \psi_5 + \psi_6 = x_1 + x_2 + x_3.$$

Another example will occur in §58. Thus we can note:

(52) **A symmetric function of values conjugate under the group $G$ belongs to $G$ or to a group containing $G$.**[2]

If a function $\xi_1$ belongs to a group $\Xi$ containing $H$ but no other permutations of $G$, it behaves under $G$ as a function belonging to $H$ does. For we can arrange the subscripts so that a permutation $t_i$ of $G$ converting

$$\psi_1 \rightarrow \psi_i$$

converts also

$$\xi_1 \rightarrow \xi_i,$$

and if

$$t_i t_j = t_k,$$

we have

$$\psi_{ij} = \psi_k$$

and also

$$\xi_{ij} = \xi_k.$$

For example, the function

$$\xi_1 = x_1 + x_2 - x_3 - x_4$$

[1] But $\psi_1\psi_2\psi_3$ is not symmetric in the $x_i$.

[2] If to a group containing $G$, the corresponding group Γ is imprimitive, by §51.

belonging to

$$\Xi = 1, (12), (34), (12)(34)$$

behaves under

$$G = 1, (23), (24), (34), (234), (243)$$

like the function

$$\psi_1 = x_2 - x_3 - x_4$$

belonging to

$$\Pi = 1, (34).$$

We have:

| (1) | (23) | (24) | (34) | (234) | (243) |
|---|---|---|---|---|---|
| $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_1$ | $\psi_2$ | $\psi_3$ |
| $\psi_2$ | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_3$ | $\psi_1$ |
| $\psi_3$ | $\psi_3$ | $\psi_1$ | $\psi_2$ | $\psi_1$ | $\psi_2$ |

,

and this remains correct if we replace $\psi$ by $\xi$. Permutations outside $G$ replace all $\psi_i$ although not all $\xi_i$; but they alter any sum of either the $\psi_i$ or the $\xi_i$. Hence we infer:

(53) **When values taken under $G$ are such that**

**(1) the sum of any values[1] conjugate under a group belongs to the group**

**(2) any value belongs to a group such that**

**(a) the group contains a subgroup but no other permutations of $G$**

**(b) $G$ contains all permutations leaving unaltered or interchanging the values**

**then the sum of the values belongs to $G$.**

The proposition is true also for other symmetric functions than sums, but is needed mostly for these.[2]

In a sense which we are apt to imply, Lagrange's Theorem fails whenever two conjugate values $\psi_i$ are equal, making the denominator in the expression for $\varphi_1$ vanish. We note however that this may not happen in case of the general equation, as will be explained in §70.

[1] Any values of those taken under $G$.

[2] Compare §§44 and 48.

## §37. PLAN OF LAGRANGE

It will be clear now why the solution of a resolvent equation may imply the solution of the general equation

$$f(x) = x^n - c_1x^{n-1} + \ldots \pm c_n = 0$$

for its roots

$$x_1, \ldots, x_n.$$

This is so because a root $x_1$ of the general equation is an $n$-valued function of the $x_i$ belonging to the subgroup

$$X_1 = S^{n-1}$$

of $S^n$ which acts on all letters $x_i$ other than $x_1$. Hence $x_1$ is by Lagrange's Theorem rationally computable from the coefficients $c_i$ of the general equation and any function of the $x_i$ that belongs to $X_1$ or a subgroup of $X_1$. But any such function is a root of a resolvent equation.

Since $X_1$ must contain identity as subgroup, a root of the general equation is certainly computable from the coefficients $c_i$ and a function

$$v = \alpha_1x_1 + \alpha_2x_2 + \ldots + \alpha_nx_n$$

that belongs to identity and hence is a root of a resolvent equation of degree $n!$.

Suppose that a series of subgroups from the symmetric group $S$ on the $n$ letters $x_i$ down to identity is

$$\begin{array}{ccccccc} S & \leftarrow j \rightarrow & G & \leftarrow k \rightarrow & H & \ldots & 1 \\ c_i & & \varphi & & \psi & & v, \end{array}$$

where the indices of the groups are written between them and the functions of the $x_i$ belonging to the groups are written below them. Then $\varphi$ is a root of a resolvent equation of degree $j$ whose coefficients are rationally computable from the $c_i$; $\psi$ is a root of a resolvent equation of degree $k$ whose coefficients are rationally computable from $\varphi$ and the $c_i$; and so on. We can construct a chain of resolvent equations

$$\varphi^j + R_1(c_i)\varphi^{j-1} + \ldots = 0$$
$$\psi^k + R_1'(\varphi, c_i)\psi^{k-1} + \ldots = 0$$
$$\ldots\ldots\ldots$$

if need be down to identity and consider these equations solved if their roots can be computed by algebraic operations. This is the plan of Lagrange.

Is our task completed? Alas, here our troubles begin! For the symmetric group on $n$ letters $x_i$ has by proposition (47) no other subgroup than the alternating group of index less than $n$ when $n$ is greater than four; and the alternating group has under such circumstances by proposition (48) no subgroup at all. It follows that among the resolvent equations is one of degree $n$ as the general equation is, and such an equation we cannot solve. Thus Lagrange's tentative plan of solving general equations fails when $n$ is greater than four.

The solution of the general biquadratic equation

$$f(x) = x^4 - c_1x^3 + c_2x^2 - c_3x + c_4 = 0$$

is possible in several ways. One plan of solution[1] is this:

$$\begin{array}{lll} & S & c_i \\ 2\{ & & \\ & A & \sqrt{\Delta} \\ 3\{ & & \\ & V & (x_1 - x_2)(x_3 - x_4) \\ 2\{ & & \\ & H & [\alpha(x_1 - x_2) + \beta(x_3 - x_4)]^2 \\ 2\{ & & \\ & 1 & \alpha(x_1 - x_2) + \beta(x_3 - x_4), \end{array}$$

where the indices of the groups are to their left and the functions belonging to the groups on their right and where[2]

$$H = 1, (12)(34).$$

Since no resolvent equation is of degree more than three, the solution of the general biquadratic equation is reduced to the solution of quadratic and cubic resolvents.

### §38. LAGRANGE'S SOLVENT

The solution of the general cubic equation

$$f(x) = x^3 + px - q = 0$$

in chapter two is based on the plan:

$$\begin{array}{lll} & S & p, q \\ 2\{ & & \\ & A & \sqrt{\Delta} \\ 3\{ & & \\ & 1 & \varphi = x_1 + \omega x_2 + \omega^2 x_3. \end{array}$$

[1] Another plan will be found in §46.

[2] $V$ was given in §31.

It is not possible to avoid a resolvent equation of degree three, and yet this resolvent solves the cubic equation which is of the same degree but not solvable without it. By what virtue then can this resolvent render such a service? Evidently because it is binomial making a solution possible by extraction of root; for it is

$$\varphi^3 = R(\sqrt{\Delta}, q, \omega),$$

the $\omega$ coming from the irrational coefficients of $\varphi$.

A sudden thought flashes through our minds. If we cannot avoid a resolvent equation of the same degree as the general equation, just when is the resolvent binomial?

If the resolvent equation

$$\psi^j - R(\varphi, c_i) = 0$$

for the values $\psi_i$ conjugate under $G$ of $\varphi$ is binomial, then

$$\psi_0 = \sqrt[j]{R},\ \psi_1 = \epsilon\sqrt[j]{R},\ \ldots,\ \psi_{j-1} = \epsilon^{j-1}\sqrt[j]{R},$$

where $\epsilon$ is the primitive root of unity[1] defined by the formula

$$\epsilon = \cos\frac{2\pi}{j} + i\sin\frac{2\pi}{j}.$$

The conjugate values $\psi_i$ differ by constant factors only and therefore belong to a normal subgroup $N$ of $G$.

A binomial resolvent either is of prime degree or can be replaced by such resolvents. For assuming that

$$j = p \cdot q$$

where $p$ and $q$ are prime, we can set

$$\psi^q = \chi$$

and

$$\chi^p = R(\varphi, c_i).$$

It follows that the group $G$ of a binomial resolvent[2] either has a normal subgroup of prime index or a series of such:

$$\begin{array}{ccccc} G & \xleftrightarrow{p} & J & \xleftrightarrow{q} & N \\ \varphi & & \chi & & \psi. \end{array}$$

Conversely,

(54) **if the group $G$ has a normal subgroup $N$ of prime index, then $G$ is the group of a binomial resolvent for $N$.**[2]

[1] Cf. §§79 and 84.

[2] Cf. §34, last two lines.

Let the index of $N$ in $G$ be a prime number $p$, and let $\psi_0$ be a function of the $x_i$ belonging to $N$. Under the partitions of

$$G = N + Nt_1 + Nt_2 + \ . \ . \ . + Nt_{p-1}$$

the conjugate values

$$\psi_0, \psi_1, \psi_2, \ . \ . \ . \ , \psi_{p-1}$$

are interchanged in $p$ different ways, as explained in §33, and by proposition (49) the permutations between these conjugate values form a group. This group is circular by proposition (25), and we may set

$$\Gamma = 1, \tau, \tau^2, \ . \ . \ . \ , \tau^{p-1}$$
$$\tau = (\psi_0\psi_1 \ . \ . \ . \ \psi_{p-1}).$$

We now form the function

$$(\epsilon,\psi)_0 = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \ . \ . \ . + \epsilon^{p-2}\psi_{p-2} + \epsilon^{p-1}\psi_{p-1}$$

also belonging to $N$, where

$$\epsilon = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}$$

is a primitive root of unity[1] and

$$\epsilon^p = 1.$$

A permutation in the partition $Nt_1$ interchanges the $\psi_i$ of the function as $\tau$ indicates, therefore cyclically, converting $(\epsilon,\psi)_0$ into

$$\begin{aligned}(\epsilon,\psi)_1 &= \psi_1 + \epsilon\psi_2 + \epsilon^2\psi_3 + \ . \ . \ . + \epsilon^{p-2}\psi_{p-1} + \epsilon^{p-1}\psi_0 \\ &= \epsilon^{-1}(\epsilon,\psi)_0,\end{aligned}$$

while a permutation in the partition $Nt_2$ interchanges the $\psi_i$ as $\tau^2$ indicates converting $(\epsilon,\psi)_0$ into

$$\begin{aligned}(\epsilon,\psi)_2 &= \psi_2 + \epsilon\psi_3 + \epsilon^2\psi_4 + \ . \ . \ . + \epsilon^{p-2}\psi_0 + \epsilon^{p-1}\psi_1 \\ &= \epsilon^{-2}(\epsilon,\psi)_0,\end{aligned}$$

and so on. It appears that the permutations of

| | | | | | |
|---|---|---|---|---|---|
| $G$ on the $x_i$: | $N$ | $Nt_1$ | $Nt_2$ | ........ | $Nt_{p-1}$ |
| like those of $\Gamma$ on the $\psi_i$: | 1 | $\tau$ | $\tau^2$ | ........ | $\tau^{p-1}$ |
| convert $(\epsilon,\psi)_0$ into: | $(\epsilon,\psi)_0$ | $(\epsilon,\psi)_1$ | $(\epsilon,\psi)_2$ | ........ | $(\epsilon,\psi)_{p-1}$ |
| equal to: | $(\epsilon,\psi)_0$ | $\epsilon^{-1}(\epsilon,\psi)_0$ | $\epsilon^{-2}(\epsilon,\psi)_0$ | ........ | $\epsilon^{-p+1}(\epsilon,\psi)_0$ |
| and to: | $(\epsilon,\psi)_0$ | $\epsilon^{p-1}(\epsilon,\psi)_0$ | $\epsilon^{p-2}(\epsilon,\psi)_0$ | ........ | $\epsilon(\epsilon,\psi)_0$ |

[1] Cf. §§79 and 84.

if we multiply by $\epsilon^p = 1$.

As we have

$$(\epsilon,\psi)_i^p = [\epsilon^{p-i}(\epsilon,\psi)_0]^p = (\epsilon,\psi)_0^p,$$

the function $(\epsilon,\psi)_0^p$ is unaltered by any permutation of $G$ and such a permutation alone, so that by Lagrange's Theorem

$$(\epsilon,\psi)_0^p = R(c_i,\varphi,\epsilon).$$

Hence the function $(\epsilon,\psi)_0$ is a root of the binomial resolvent

$$r(\epsilon,\psi) = (\epsilon,\psi)^p - R(c_i,\varphi,\epsilon) = 0,$$

and the group $G$ satisfies the proposition.

The functions $(\epsilon,\psi)_i$ permit the calculation of the $\psi_i$, as it was illustrated in the solution of the general cubic equation of chapter two by the functions $(\omega,x)_i$, and such a function $(\epsilon,\psi)_i$ is called **Lagrange's solvent**[1] for a normal subgroup $N$ of $G$.

It follows that

(55) **a resolvent equation for a normal subgroup of prime index is binomial if constructed on Lagrange's solvent for that subgroup.**

It should be noticed that not every function $\psi_i$ belonging to $N$ has its $p$-th power belong to $G$ as Lagrange's solvent $(\epsilon,\psi)_i$ has.

## §39. SPECIAL CASE OF SOLVENTS

For the alternating group of index

$$p = 2$$

in the symmetric group and

$$\epsilon = -1$$

we have by proposition (40)

$$\psi_1 = S_1 + S_2\sqrt{\Delta}$$
$$\psi_2 = S_1 - S_2\sqrt{\Delta},$$

so that Lagrange's solvent is

$$(\epsilon,\psi) = \psi_1 + \epsilon\psi_2 = 2S_2\sqrt{\Delta},$$

in the simplest case just $\sqrt{\Delta}$.

[1] Also called Lagrange's resolvent (function).

Reviewing the solution of the general cubic equation in chapter two, we observe that Lagrange's solvent for the alternating group complies with the rules, but Lagrange's solvent

$$(\omega,x) = x_1 + \omega x_2 + \omega^2 x_3$$

for identity in the alternating group is constructed on the functions

$$\begin{aligned} &x_1 \text{ with group } 1, (23) \\ &x_2 \text{ with group } 1, (13) \\ &x_3 \text{ with group } 1, (12) \end{aligned}$$

which do not belong to identity. This is possible because the functions $x_i$ belong to groups containing identity but no other permutations of the alternating group

$$A = 1, (123), (132),$$

and because there are no permutations outside the alternating group that leave the functions $x_i$ unaltered.[1]

The significance of the last condition is seen on the function

$$(\omega,\varphi) = \varphi_1 + \omega\varphi_2 + \omega^2\varphi_3$$

with

$$\begin{aligned} \varphi_1 &= x_1x_2 + x_3x_4 \\ \varphi_2 &= x_1x_3 + x_2x_4 \\ \varphi_3 &= x_1x_4 + x_2x_3, \end{aligned}$$

which belong to groups[2] containing identity and no other permutations of

$$G = 1, (123), (132)$$

but are unaltered by the permutations of $V$,[3] which are not in $G$. The function $(\omega,\varphi)$ behaves under $G$ as if it were Lagrange's solvent for identity in four letters $x_i$:

| | |
|---|---|
| 1 | $\varphi_1 + \omega\varphi_2 + \omega^2\varphi_3 = (\omega,\varphi)$ |
| (132) | $\varphi_2 + \omega\varphi_3 + \omega^2\varphi_1 = \omega^{-1}(\omega,\varphi)$ |
| (123) | $\varphi_3 + \omega\varphi_1 + \omega^2\varphi_2 = \omega^{-2}(\omega,\varphi)$ |

,

[1] Compare proposition (53).

[2] These groups were given in §19.

[3] $V$ was given in §31.

yet this function belongs not to identity but to the group

$$\{1,V\} = V,$$

and its cube not to $G$ but to the group

$$\{G,V\} = A^4.$$

It appears that the functions $\psi_i$ of Lagrange's solvent may conditionally be replaced by functions $\xi_i$:

(56) **Lagrange's solvents for a normal subgroup $N$ of $G$ can be constructed on conjugate functions belonging to $N$, or on conjugate functions belonging to groups containing $N$ but no other permutations of $G$ if there are no permutations outside $G$ that leave the conjugate functions unaltered.**

## §40. LIMITS OF LAGRANGE'S PLAN

We have learned how to construct binomial resolvents. Does it help us to solve the general equation of degree $n$? Only when the composition-factors of the symmetric group on $n$ letters $x_i$ are prime.

**Whenever the symmetric group in $n$ letters $x_i$ has a series of subgroups each normal and of prime index in the preceding, the series beginning with the symmetric group and ending with identity, then we can solve the general equation of degree $n$ by algebraic operations using primitive roots of unity.**[1]

But when these conditions do not hold, it would not seem possible to solve the general equation of degree $n$ by algebraic operations, which beside the rational operations include the extraction of roots.[2]

With scarcely any hope left we therefore acknowledge failure, for we have no such series of normal subgroups with prime indices if $n$ is greater than four, since the symmetric group then has no other normal subgroup than the alternating group and the

[1] We need primitive roots of unity for Lagrange's solvents, but they can be computed, as will be explained in §84.

[2] Compare the statement of our conclusion in §§69 and 76. For the final statement see §82.

alternating group no normal subgroup at all. The composition-series for

$$n > 4$$

is

$$S \leftarrow 2 \rightarrow A \leftarrow \tfrac{n!}{2} \rightarrow 1,$$

where $n!/2$ is not prime.

This seems to put an end to all attempts of solving the general equation of degree higher than four: indeed we shall find[1] that it is not solvable. While special equations of higher degree can be solved, they elude the grip of Lagrange.

That they do, is not a fault. Lagrange's plan of solving equations has been treated as inferior: that it is not. Directed toward the solution of the general equation, it is perfect as such; and concluding our study of it, we salute the genius of the master.

[1] Cf. §76.

# CHAPTER VIII

## GENERAL EQUATIONS

### A. QUADRATIC EQUATION:

$$\boxed{x^2 - c_1 x + c_2 = 0}$$

§41. The binomial resolvent is given by the plan

$$\begin{array}{ccc} S & \xleftarrow{\ \ 2\ \ }\!\!\rightarrow & A = 1 \\ c_i, \Delta & & \sqrt{\Delta}, \end{array}$$

where the symmetric group is

$$S = 1, (12)$$

and the alternating group, normal in the symmetric, shrinks to identity.

Lagrange's solvent for

$$A = 1$$

is[1]

$$\sqrt{\Delta} = x_1 - x_2.$$

Its square, belonging to the symmetric group and hence by Lagrange's Theorem rationally computable from the $c_i$, is the discriminant

$$\begin{aligned} \Delta &= (x_1 - x_2)^2 \\ &= (x_1 + x_2)^2 - 4x_1x_2 \\ &= c_1^2 - 4c_2. \end{aligned}$$

This gives us the binomial resolvent for $A$.

The roots $x_1$ and $x_2$, belonging as functions of the $x_i$ to the group $A = 1$, are by Lagrange's Theorem rationally computable from

$$\sqrt{\Delta} = \sqrt{c_1^2 - 4c_2}.$$

We set

$$\begin{aligned} x_1 + x_2 &= c_1 \\ \sqrt{\Delta} = x_1 - x_2 &= \sqrt{c_1^2 - 4c_2} \end{aligned}$$

[1] Cf. §39.

and have

$$x_1 = \frac{c_1}{2} + \sqrt{\left(\frac{c_1}{2}\right)^2 - c_2}$$

$$x_2 = \frac{c_1}{2} - \sqrt{\left(\frac{c_1}{2}\right)^2 - c_2}.$$

If we write the quadratic equation in the binomial form:

$$a_0x^2 + 2a_1x + a_2 = 0,$$

and then substitute from

$$c_1 = -\frac{2a_1}{a_0}$$

$$c_2 = \frac{a_2}{a_0},$$

we obtain

$$x_1 = \frac{-a_1 + \sqrt{a_1^2 - a_0a_2}}{a_0}$$

$$x_2 = \frac{-a_1 - \sqrt{a_1^2 - a_0a_2}}{a_0}.$$

### B. CUBIC EQUATION:

$$\boxed{x^3 - c_1x^2 + c_2x - c_3 = 0}$$

§42. The solution by binomial resolvents is given with the plan

$$\begin{array}{ccccc} S & \xleftrightarrow{2} & A & \xleftrightarrow{3} & 1 \\ c_i, \Delta & & \sqrt{\Delta}, \varphi_i^3 & & \varphi_i \end{array}$$

where

$$S = 1, (123), (132), (12), (13), (23)$$

and

$$A = 1, (123), (132).$$

The square of Lagrange's solvent for $A$ is the discriminant[1]

$$\Delta = (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2$$

which is symmetric in the $x_i$ and by Lagrange's Theorem rationally computable from the $c_i$. To do the computation,[2] we notice that the discriminant is homogeneous of degree six in all the $x_i$ and of degree not more than four in any one of them. Hence it is expressible as

$$\Delta = \sum c_1^{\nu_1}c_2^{\nu_2}c_3^{\nu_3},$$

[1] Cf. §39.

[2] Cf. §24, end.

where the weight of each term is

$$W = \nu_1 + 2\nu_2 + 3\nu_3 = 6,$$

while the total degree is

$$D = \nu_1 + \nu_2 + \nu_3 \leqq 4.$$

The combinations of the $\nu_i$ satisfying these conditions are

| $\nu_1$ | $\nu_2$ | $\nu_3$ |
|---|---|---|
| 0 | 0 | 2 |
| 0 | 3 | 0 |
| 1 | 1 | 1 |
| 2 | 2 | 0 |
| 3 | 0 | 1. |

Consequently

$$\Delta = l_1c_3^2 + l_2c_2^3 + l_3c_1c_2c_3 + l_4c_1^2c_2^2 + l_5c_1^3c_3,$$

where the $l_i$ are numerical coefficients.

We proceed to compute these coefficients with the help of special equations. Thus the equation

(1) $$x^3 - x = 0$$

with

$$x_1 = 0,\ x_2 = 1,\ x_3 = -1$$

and

$$c_1 = 0,\ c_2 = -1,\ c_3 = 0$$

gives

$$\Delta = -l_2$$

from the equation for $\Delta$, and gives directly

$$\Delta = (0 - 1)^2(0 + 1)^2(1 + 1)^2 = 4,$$

whence

$$l_2 = -4.$$

(2) $$x^3 - 2x^2 + x = 0$$

with

$$x_1 = 0,\ x_2 = 1,\ x_3 = 1$$

and

$$c_1 = 2,\ c_2 = 1,\ c_3 = 0$$

gives

$$\Delta = l_2 + 4l_4 = 4l_4 - 4 \qquad [l_2 = -4$$

and

$$\Delta = (0 - 1)^2(0 - 1)^2(1 - 1)^2 = 0,$$

whence

$$l_4 = 1.$$

(3) $x^3 - 3x + 2 = 0$

with

$$x_1 = 1, x_2 = 1, x_3 = -2$$

gives

$$\Delta = 4l_1 - 27l_2 = 0,$$

whence

$$l_1 = -27.$$

(4) $x^3 - 3x^2 + 4 = 0$

with

$$x_1 = 2, x_2 = 2, x_3 = -1$$

gives

$$\Delta = 16l_1 - 108l_5 = 0, \qquad [l_1 = -27$$

whence

$$l_5 = -4.$$

(5) $x^3 - x^2 - x + 1 = 0$

with

$$x_1 = 1, x_2 = 1, x_3 = -1$$

gives

$$\Delta = l_1 - l_2 + l_3 + l_4 - l_5 = 0, [l_4 = 1, \; l_5 = -4$$

whence

$$l_3 = 18.$$

Substituting the numerical values of the $l_\iota$, we have as binomial resolvent for $A$ the equation

$$\Delta = -27c_3^2 - 4c_2^3 + 18c_1c_2c_3 + c_1^2c_2^2 - 4c_1^3c_3.$$

§43. As Lagrange's solvent for identity we can take by proposition (56) the function

$$(\omega,x) = x_1 + \omega x_2 + \omega^2 x_3 = \varphi_1.$$

Its cube belongs to the alternating group and is by Lagrange's Theorem rationally computable from $\sqrt{\Delta}$: as alternating function it is by proposition (40) of the form

$$\varphi_1^3 = S_1 + S_2\sqrt{\Delta}.$$

To compute it, we have

$$\begin{aligned}\varphi_1^3 &= (x_1 + \omega x_2 + \omega^2 x_3)^3 \\ &= x_1^3 + x_2^3 + x_3^3 + 3\omega(x_1^2x_2 + x_2^2x_3 + x_3^2x_1) \\ &\qquad + 3\omega^2(x_1x_2^2 + x_2x_3^2 + x_3x_1^2) + 6x_1x_2x_3.\end{aligned}$$

Substituting from

$$\omega = \frac{-1 + \sqrt{-3}}{2}$$

$$\omega^2 = \frac{-1 - \sqrt{-3}}{2}$$

and then setting[1]

$$\begin{aligned} s(x_1^3) &= x_1^3 + x_2^3 + x_3^3 \\ s(x_1^2x_2) &= x_1^2x_2 + x_2^2x_3 + x_3^2x_1 + x_1x_2^2 + x_2x_3^2 + x_3x_1^2 \\ s(x_1x_2x_3) &= x_1x_2x_3 \end{aligned}$$

we obtain

$$\varphi_1^3 = s(x_1^3) - \frac{3}{2}s(x_1^2x_2) + \frac{3\sqrt{-3}}{2}(x_1^2x_2 + x_2^2x_3 + x_3^2x_1 - x_1x_2^2 - x_2x_3^2 - x_3x_1^2) + 6s(x_1x_2x_3).$$

Since the third term alone is non-symmetric in the $x_i$, it must contain $\sqrt{\Delta}$: indeed we find

$$\begin{aligned} \sqrt{\Delta}_{i<k} &= (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \\ &= x_1^2x_2 + x_2^2x_3 + x_3^2x_1 - x_1x_2^2 - x_2x_3^2 - x_3x_1^2, \end{aligned}$$

so that

$$\varphi_1^3 = s(x_1^3) - \frac{3}{2}s(x_1^2x_2) + 6s(x_1x_2x_3) + \frac{3\sqrt{-3}}{2}\sqrt{\Delta}_{i<k}.$$

Computing now by the method used in §42:

$$\begin{aligned} s(x_1^3) &= 3c_3 - 3c_1c_2 + c_1^3 \\ s(x_1^2x_2) &= -3c_3 + c_1c_2 \\ s(x_1x_2x_3) &= c_3, \end{aligned}$$

or computing directly:

$$\begin{aligned} \varphi_1^3 &= x_1^3 + x_2^3 + x_3^3 + 6x_1x_2x_3 - \frac{3}{2}(x_1^2x_2 + x_2^2x_3 + x_3^2x_1 \\ &\qquad + x_1x_2^2 + x_2x_3^2 + x_3x_1^2) + \frac{3\sqrt{-3\Delta}}{2} \\ &= (x_1 + x_2 + x_3)^3 - \frac{9}{2}(x_1^2x_2 + x_2^2x_3 + x_3^2x_1 + x_1x_2^2 \\ &\qquad + x_2x_3^2 + x_3x_1^2) + \frac{3\sqrt{-3\Delta}}{2} \\ &= (x_1 + x_2 + x_3)^3 - \frac{9}{2}(x_1x_2 + x_2x_3 + x_3x_1)(x_1 + x_2 + x_3) \\ &\qquad + \frac{27}{2}x_1x_2x_3 + \frac{3\sqrt{-3\Delta}}{2}, \end{aligned}$$

[1] Cf. §22.

we have as binomial resolvent for the function

$$\varphi_1 = x_1 + \omega x_2 + \omega^2 x_3$$

belonging to identity the equation

$$\varphi_1^3 = \frac{27}{2}c_3 - \frac{9}{2}c_1c_2 + c_1^3 + \frac{3\sqrt{-3\Delta}}{2}$$
$$= \frac{1}{2}(27\, c_3 - 9c_1c_2 + 2c_1^3 + 3\sqrt{-3\Delta}).$$
$$i < k$$

Applying the transposition (23), we find as binomial resolvent for the function

$$\varphi_2 = x_1 + \omega^2 x_2 + \omega x_3$$

the equation

$$\varphi_2^3 = \frac{1}{2}(27c_3 - 9c_1c_2 + 2c_1^3 - 3\sqrt{-3\Delta}),$$
$$i < k$$

since a transposition changes the sign of $\sqrt{\Delta}$.

The conjugate values of $\varphi_1$ under $A$ are

$$\varphi_3 = \omega^2\varphi_1$$
$$\varphi_5 = \omega\varphi_1,$$

the conjugate values of $\varphi_2$ under $A$ are

$$\varphi_4 = \omega\varphi_2$$
$$\varphi_6 = \omega^2\varphi_2;$$

and these conjugate values are the remaining roots of the resolvent equations.

§44. We have found the value of every function $\varphi_i$ belonging to identity, and all what remains for us to do is to compute the value of every function $x_i$: in terms of any $\varphi_i$ if we please, by theorem and method of Lagrange in chapter seven, or more conveniently from the sum of two $\varphi_i$. For the function $\varphi_1$ takes under the group

$$X_1 = 1\ (23)$$

of $x_1$ the conjugate value $\varphi_2$, and the function

$$\varphi_1 + \varphi_2$$

lends itself by proposition (53) to a rational computation of $x_1$. Recalling that

$$\omega + \omega^2 = -1,$$

we have

$$\begin{array}{l} x_1 + \omega x_2 + \omega^2 x_3 = \varphi_1 \\ x_1 + \omega^2 x_2 + \omega x_3 = \varphi_2 \\ \hline 2\,x_1 - x_2 - x_3 = \varphi_1 + \varphi_2 \\ x_1 + x_2 + x_3 = c_1 \\ \hline 3\,x_1 = c_1 + \varphi_1 + \varphi_2. \end{array}$$

Similarly we may compute any $x_i$ in terms of $\varphi_i + \varphi_k$ properly chosen. Or we may find the $x_i$ adding the same equations as they stand and multiplied by

$$\omega^2,\ \omega,\ 1$$

and then by

$$\omega,\ \omega^2,\ 1$$

respectively. Recalling that

$$1 + \omega + \omega^2 = 0,$$

we thus have

$$\begin{array}{l} x_1 + \omega x_2 + \omega^2 x_3 = \varphi_1 \\ x_1 + \omega^2 x_2 + \omega x_3 = \varphi_2 \\ x_1 + x_2 + x_3 = c_1 \\ \hline 3x_1 = c_1 + \varphi_1 + \varphi_2 \\ 3x_2 = c_1 + \omega^2\varphi_1 + \omega\varphi_2 \\ 3x_3 = c_1 + \omega\varphi_1 + \omega^2\varphi_2, \end{array}$$

where we may substitute the proper $\varphi_i$ to rid the result of $\omega$.

§45. If the cubic equation is written in the binomial form:

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0,$$

we set

$$c_1 = -\frac{3a_1}{a_0},\ c_2 = \frac{3a_2}{a_0},\ c_3 = -\frac{a_3}{a_0}$$

and obtain

$$\begin{aligned} \Delta &= -\frac{27}{a_0^4}(a_0^2a_3^2 + 4a_0a_2^3 - 6a_0a_1a_2a_3 - 3a_1^2a_2^2 + 4a_1^3a_3) \\ &= -\frac{27}{a_0^4}\cdot\frac{(a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3)^2 - 4(a_1^2 - a_0a_2)^3}{a_0^2} \end{aligned}$$

$$\varphi_1^3 = -\frac{27}{2a_0^3}(a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3) + \frac{3\sqrt{-3\Delta}}{2}.$$

Introducing the notation

$$G_2 = a_0a_2 - a_1^2 = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}$$

$$G_3 = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$$

for functions whose virtues are revealed in the theory of invariants, we finally have

$$\Delta = -\frac{27}{a_0^6}(G_3^2 + 4G_2^3)$$

$$\varphi_1^3 = -\frac{27}{2a_0^3}(G_3 + \sqrt{G_3^2 + 4G_2^3})$$

$$\varphi_2^3 = -\frac{27}{2a_0^3}(G_3 - \sqrt{G_3^2 + 4G_2^3}),$$

and as solution of the cubic equation

$$x_1 = -\frac{a_1}{a_0} - \frac{1}{a_0\sqrt[3]{2}}(G_3 + \sqrt{G_3^2 + 4G_2^3}) - \frac{1}{a_0\sqrt[3]{2}}(G_3 - \sqrt{G_3^2 + 4G_2^3})$$

$$x_2 = -\frac{a_1}{a_0} - \frac{\omega^2}{a_0\sqrt[3]{2}}(G_3 + \sqrt{G_3^2 + 4G_2^3}) - \frac{\omega}{a_0\sqrt[3]{2}}(G_3 - \sqrt{G_3^2 + 4G_2^3})$$

$$x_3 = -\frac{a_1}{a_0} - \frac{\omega}{a_0\sqrt[3]{2}}(G_3 + \sqrt{G_3^2 + 4G_2^3}) - \frac{\omega^2}{a_0\sqrt[3]{2}}(G_3 - \sqrt{G_3^2 + 4G_2^3}).$$

Example:

$$x^3 - 7x^2 + 14x - 8 = 0,$$

$$a_0 = 1,\ a_1 = -\frac{7}{3},\ a_2 = \frac{14}{3},\ a_3 = -8.$$

$$G_2 = a_0a_2 - a_1^2 = -\frac{7}{9}$$

$$G_3 = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3 = -\frac{20}{27}$$

$$\varphi_1^3 = -\frac{27}{2}(G_3 + \sqrt{G_3^2 + 4G_2^3}) = 10 - 9\sqrt{-3}$$

$$\varphi_1 = -2 - \sqrt{-3}$$

$$\varphi_2 = -2 + \sqrt{-3}$$

$$x_1 + \omega x_2 + \omega^2 x_3 = -2 - \sqrt{-3}$$

$$x_1 + \omega^2 x_2 + \omega x_3 = -2 + \sqrt{-3}$$

$$x_1 + x_2 + x_3 = 7$$

$$x_1 = 1$$

$$x_2 = 2 \qquad [\omega + \omega^2 = -1;\ \omega - \omega^2 = \sqrt{-3}$$

$$x_3 = 4.$$

## C. BIQUADRATIC EQUATION:

$$\boxed{x^4 - c_1x^3 + c_2x^2 - c_3x + c_4 = 0}$$

§46. The biquadratic equation written in the binomial form is

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0,$$

where

$$c_1 = -\frac{4a_1}{a_0}, \; c_2 = \frac{6a_2}{a_0}, \; c_3 = -\frac{4a_3}{a_0}, \; c_4 = \frac{a_4}{a_0}.$$

A convenient solution[1] is given by the plan

$$\begin{array}{ccccc} S & \xleftrightarrow{3} & G & \xleftrightarrow{2} & N \\ c_i & & y_1, \xi_i^2 & & \xi_i, \end{array}$$

where $S$ is the symmetric group on four letters $x_i$, as given in §21, while

$$G = 1, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)$$

and

$$N = 1, (12), (34), (12)(34).$$

A kind fate thus spared us the search for the discriminant.

Since the group $G$ is not normal in $S$, the cubic resolvent for $y_1$ is not binomial and has to be solved by one square and one cube root, as shown under B.

Let[2]

$$y_1 = x_1x_2 + x_3x_4.$$

It is a root of the resolvent equation

$$\begin{aligned}(y - y_1)(y - y_2)(y - y_3) \\ &= y^3 - (y_1 + y_2 + y_3)y^2 + (y_1y_2 + y_2y_3 + y_3y_1)y - y_1y_2y_3 \\ &= A_0y^3 + 3A_1y^2 + 3A_2y + A_3 = 0,\end{aligned}$$

whose other roots are the conjugate values

$$\begin{aligned} y_2 &= x_1x_3 + x_2x_4 \\ y_3 &= x_1x_4 + x_2x_3 \end{aligned}$$

of $y_1$ under $S$. The coefficients of this equation are symmetric in the $y_i$. As a permutation on the $x_i$ can do no more than interchange the $y_i$, the coefficients are symmetric also in the $x_i$ and hence rationally computable from the $c_i$.

[1] Cf. §37.
[2] Cf. §§19 and 39.

Noting that

$$A_0 = 1,$$

we compute:

$$-3A_1 = y_1 + y_2 + y_3$$
$$= x_1x_2 + x_3x_4 + x_1x_3 + x_2x_4 + x_1x_4 + x_2x_3 = c_2 = \frac{6a_2}{a_0}.$$

Further:

$$3A_2 = y_1y_2 + y_2y_3 + y_3y_1$$
$$= (x_1x_2 + x_3x_4)(x_1x_3 + x_2x_4) + \dots = S(x_i),$$

where any term of $S(x_i)$ is of degree four in all the $x_i$ and of degree not more than two in any one $x_i$. Hence[1]

$$S(x_i) = \sum c_1^{\nu_1}c_2^{\nu_2}c_3^{\nu_3}c_4^{\nu_4},$$

with the condition that

$$W = \nu_1 + 2\nu_2 + 3\nu_3 + 4\nu_4 = 4$$
$$D = \nu_1 + \nu_2 + \nu_3 + \nu_4 \leqq 2.$$

The possible combinations of the $\nu_i$ are:

| $\nu_1$ | $\nu_2$ | $\nu_3$ | $\nu_4$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 0 | 2 | 0 | 0, |

and we have:

$$3A_2 = l_1c_4 + l_2c_1c_3 + l_3c_2^2.$$

To compute the $l_i$, we use special values of the $x_i$ and the corresponding values of the $c_i$ and the $y_i$, as we did in §42:

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $c_4$ | $c_1c_3$ | $c_2^2$ | $y_1$ | $y_2$ | $y_3$ | $3A_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | −1 | −1 | 1 | 0 | | 2 | −2 | −2 | −4 |
| 1 | 1 | −1 | 0 | 0 | −1 | | 1 | −1 | −1 | −1 |

The first set of values gives

$$3A_2 = l_1c_4 + l_2c_1c_3 + l_3c_2^2 = l_3$$

[1] Cf. §24, end.

and

$$3A_2 = y_1y_2 + y_2y_3 + y_3y_1 = 0,$$

whence

$$l_3 = 0.$$

The second:

$$3A_2 = l_1c_4 + l_2c_1c_3 = l_1 \qquad [l_3 = 0$$

and

$$3A_2 = y_1y_2 + y_2y_3 + y_3y_1 = -4,$$

whence

$$l_1 = -4.$$

The third:

$$3A_2 = l_1c_4 + l_2c_1c_3 = -l_2$$

and

$$3A_2 = y_1y_2 + y_2y_3 + y_3y_1 = -1,$$

whence

$$l_2 = 1.$$

Thus we have

$$3A_2 = -4c_4 + c_1c_3 = \frac{-4a_0a_4 + 16a_1a_3}{a_0^2}.$$

Likewise,

$$-A_3 = y_1y_2y_3 = (x_1x_2 + x_3x_4)(x_1x_3 + x_2x_4)(x_1x_4 + x_2x_3)$$
$$= \sum c_1^{\nu_1}c_2^{\nu_2}c_3^{\nu_3}c_4^{\nu_4},$$

with the condition that

$$W = \nu_1 + 2\nu_2 + 3\nu_3 + 4\nu_4 = 6$$
$$D = \nu_1 + \nu_2 + \nu_3 + \nu_4 \leqq 3.$$

The possible combinations of the $\nu_i$ are:

| $\nu_1$ | $\nu_2$ | $\nu_3$ | $\nu_4$ |
|---|---|---|---|
| 0 | 1 | 0 | 1 |
| 0 | 0 | 2 | 0 |
| 0 | 3 | 0 | 0 |
| 1 | 1 | 1 | 0 |
| 2 | 0 | 0 | 1, |

and we have

$$-A_3 = l_1c_2c_4 + l_2c_3^2 + l_3c_2^3 + l_4c_1c_2c_3 + l_5c_1^2c_4.$$

Special values of the $x_i$ give:

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $c_2c_4$ | $c_3^2$ | $c_2^3$ | $c_1c_2c_3$ | $c_1^2c_4$ | $y_1$ | $y_2$ | $y_3$ | $-A_3$ | Result |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | | 0 | $l_3 = 0$ |
| 1 | 1 | $-1$ | $-1$ | $-2$ | 0 | | 0 | 0 | 2 | $-2$ | $-2$ | 8 | $l_1 = -4$ |
| 1 | 1 | $-2$ | 0 | 0 | 4 | | 0 | 0 | 1 | $-2$ | $-2$ | 4 | $l_2 = 1$ |
| 1 | 1 | $-1$ | 0 | 0 | 1 | | 1 | 0 | 1 | $-1$ | $-1$ | 1 | $l_4 = 0$ |
| 1 | $-1$ | $-1$ | $-1$ | 0 | 4 | | | $-4$ | 0 | | | 0 | $l_5 = 1$ |

Thus we have

$$-A_3 = -4c_2c_4 + c_3^2 + c_1^2c_4 = \frac{-24a_0a_2a_4 + 16a_0a_3^2 + 16a_1^2a_4}{a_0^3}.$$

§47. We now solve the resolvent equation

$$A_0y^3 + 3A_1y^2 + 3A_2y + A_3$$
$$= y^3 - \frac{6a_2}{a_0}y^2 - \frac{4a_0a_4 - 16a_1a_3}{a_0^2}y + \frac{24a_0a_2a_4 - 16a_0a_3^2 - 16a_1^2a_4}{a_0^3} = 0$$

as explained under B:

$$G_2 = A_0A_2 - A_1^2$$
$$G_3 = A_0^2A_3 - 3A_0A_1A_2 + 2A_1^3,$$

where

$$A_0 = 1, \quad A_1 = \frac{-2a_2}{a_0}$$
$$A_2 = \frac{-4a_0a_4 + 16a_1a_3}{3a_0^2}$$
$$A_3 = \frac{24a_0a_2a_4 - 16a_0a_3^2 - 16a_1^2a_4}{a_0^3}.$$

Substituting for the $A_i$, we obtain

$$G_2 = -\frac{4}{3a_0^2}(a_0a_4 - 4a_1a_3 + 3a_2^2) = -\frac{4}{3a_0^2}g_2$$
$$G_3 = \frac{16}{a_0^3}(a_0a_2a_4 - a_0a_3^2 - a_1^2a_4 + 2a_1a_2a_3 - a_2^3) = \frac{16}{a_0^3}g_3,$$

where

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2$$
$$g_3 = a_0a_2a_4 - a_0a_3^2 - a_1^2a_4 + 2a_1a_2a_3 - a_2^3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}$$

The cube of the function[1]

$$\varphi_1 = y_1 + \omega y_2 + \omega^2 y_3$$

is

$$\begin{aligned}
\varphi_1^3 &= -\frac{27}{2A_0^3}(G_3 + \sqrt{G_3^2 + 4G_2^3}) \\
&= -\frac{27}{2A_0^3}\left(\frac{16}{a_0^3}g_3 + \sqrt{\frac{256}{a_0^6}g_3^2 - \frac{256}{27a_0^6}g_2^3}\right) \\
&= \frac{-216g_3 - 24\sqrt{81g_3^2 - 3g_2^3}}{a_0^3},
\end{aligned}$$

while the cube of the function

$$\varphi_2 = y_1 + \omega^2 y_2 + \omega y_3$$

is

$$\varphi_2^3 = \frac{-216g_3 + 24\sqrt{81g_3^2 - 3g_2^3}}{a_0^3}.$$

Hence

$$\begin{aligned}
\varphi_1 &= \frac{2}{a_0}(-27g_3 - 3\sqrt{81g_3^2 - 3g_2^3})^{1/3} \\
\varphi_2 &= \frac{2}{a_0}(-27g_3 + 3\sqrt{81g_3^2 - 3g_2^3})^{1/3},
\end{aligned}$$

and we have as solution of the resolvent equation

$$\begin{aligned}
y_1 &= \frac{2a_2}{a_0} + \frac{1}{3}(\varphi_1 + \varphi_2) \\
y_2 &= \frac{2a_2}{a_0} + \frac{1}{3}(\omega^2\varphi_1 + \omega\varphi_2) \\
y_3 &= \frac{2a_2}{a_0} + \frac{1}{3}(\omega\varphi_1 + \omega^2\varphi_2).
\end{aligned}$$

The discriminant of the resolvent equation is

$$\begin{aligned}
\Delta_y &= (y_1 - y_2)^2(y_1 - y_3)^2(y_2 - y_3)^2 \\
&= (x_1x_2 + x_3x_4 - x_1x_3 - x_2x_4)^2(x_1x_2 + x_3x_4 - x_1x_4 - x_2x_3)^2 \\
&\qquad \cdot (x_1x_3 + x_2x_4 - x_1x_4 - x_2x_3)^2 \\
&= (x_1 - x_4)^2(x_2 - x_3)^2(x_1 - x_3)^2(x_2 - x_4)^2(x_1 - x_2)^2 \\
&\qquad \cdot (x_3 - x_4)^2;
\end{aligned}$$

[1] Cf. §39.

it is identical with the discriminant $\Delta_x$ of the biquadratic equation, which we thus find incidentally, and by §45 we compute:

$$\begin{aligned}\Delta &= -\frac{27}{A_0^6}(G_3^2 + 4G_2^3)\\ &= -27\left(\frac{256g_3^2}{a_0^6} - \frac{256g_2^3}{27a_0^6}\right)\\ &= \frac{256}{a_0^6}(g_2^3 - 27g_3^2).\end{aligned}$$

This permits to set

$$\begin{aligned}\varphi_1^3 &= -\frac{216g_3}{a_0^3} - \frac{3}{2}\sqrt{-3\Delta}\\ \varphi_2^3 &= -\frac{216g_3}{a_0^3} + \frac{3}{2}\sqrt{-3\Delta}.\end{aligned}$$

The resolvent equation may be brought into the form

$$\left(y - \frac{2a_2}{a_0}\right)^3 - \frac{4g_2}{a_0^2}\left(y - \frac{2a_2}{a_0}\right) + \frac{16g_3}{a_0^3} = 0,$$

which is the same as

$$z^3 + 3G_2z + G_3 = 0$$

with

$$z = y - \frac{2a_2}{a_0}.$$

Substituting

$$\frac{2z}{a_0} = y - \frac{2a_2}{a_0},$$

we reduce the resolvent equation to

$$z^3 - g_2z + 2g_3 = 0,$$

and substituting

$$-\frac{4z}{a_0} = y - \frac{2a_2}{a_0},$$

we reduce it to

$$4z^3 - g_2z - g_3 = 0,$$

historic resolvents of the biquadratic equation.

§48. As Lagrange's solvent for the normal subgroup

$$N = 1,\ (12),\ (34),\ (12)(34)$$

of $G$ we take the function

$$(\epsilon, \psi) = \psi_1 + \epsilon\psi_2 = \xi_1,$$

where

$$\epsilon = -1$$
$$\psi_1 = x_1 + x_2.$$

Belonging to $N$, $\psi_1$ takes under $G$ the conjugate value

$$\psi_2 = x_3 + x_4,$$

so that

$$\xi_1 = x_1 + x_2 - x_3 - x_4.$$

The square of $\xi_1$ belongs to $G$ and is by Lagrange's Theorem rationally computable from $y_1$:

$$\begin{aligned}
\xi_1^2 &= (x_1 + x_2 - x_3 - x_4)^2 \\
&= (x_1 + x_2 + x_3 + x_4)^2 - 4(x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4) \\
&= (x_1 + x_2 + x_3 + x_4)^2 - 4(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 \\
&\qquad\qquad + x_2x_4 + x_3x_4) + 4(x_1x_2 + x_3x_4) \\
&= \left(-\frac{4a_1}{a_0}\right)^2 - 4\frac{6a_2}{a_0} + 4y_1 \\
&= \frac{4}{a_0^2}(4a_1^2 - 6a_0a_2 + a_0^2y_1).
\end{aligned}$$

Solving this binomial resolvent for $\xi_1$, we obtain

$$\xi_1 = x_1 + x_2 - x_3 - x_4 = \pm\frac{2}{a_0}\sqrt{4a_1^2 - 6a_0a_2 + a_0^2y_1}.$$

Although the group $N$ of $\xi_1$ is not a subgroup of the group

$$X_1 = 1, (23), (24), (34), (234), (243)$$

of $x_1$, we need not go any further since it contains such a subgroup

$$H = 1, (34)$$

and satisfies both conditions set forth in proposition (53). Hence the sum of the conjugate values that $\xi_1$ takes under $X_1$ lends itself to a rational computation of $x_1$.

The equation for $\xi_1$ is converted into

$$\xi_2 = x_1 - x_2 + x_3 - x_4 = \pm\frac{2}{a_0}\sqrt{4a_1^2 - 6a_0a_2 + a_0^2y_2}$$

by the permutation (23) of $X_1$ and into

$$\xi_3 = x_1 - x_2 - x_3 + x_4 = \pm\frac{2}{a_0}\sqrt{4a_1^2 - 6a_0a_2 + a_0^2y_3}$$

by the permutation (24) of $X_1$. The other three values of $\xi_i$ under the symmetric group on the $x_i$ are the negatives of the values obtained, and from the sum of three $\xi_i$ properly chosen every $x_i$ can be rationally computed.

If we take

$$\begin{aligned} x_1 + x_2 - x_3 - x_4 &= \xi_1 \\ x_1 - x_2 + x_3 - x_4 &= \xi_2 \\ x_1 - x_2 - x_3 + x_4 &= \xi_3 \\ x_1 + x_2 + x_3 + x_4 &= -\frac{4a_1}{a_0}, \end{aligned}$$

we have as solution of the biquadratic equation

$$\begin{aligned} x_1 &= -\frac{a_1}{a_0} + \frac{1}{4}(\xi_1 + \xi_2 + \xi_3) \\ x_2 &= -\frac{a_1}{a_0} + \frac{1}{4}(\xi_1 - \xi_2 - \xi_3) \\ x_3 &= -\frac{a_1}{a_0} + \frac{1}{4}(-\xi_1 + \xi_2 - \xi_3) \\ x_4 &= -\frac{a_1}{a_0} + \frac{1}{4}(-\xi_1 - \xi_2 + \xi_3), \end{aligned}$$

where we may replace any negative $\xi_i$ by a positive one if we choose to express an $x_i$ in terms of a sum.

To determine what sign to select for the value of a $\xi_i$, we notice that any transposition between the $x_i$ leaves one of the three $\xi_i$ unaltered and interchanges, with possible change of sign, the other two. Hence the product $\xi_1\xi_2\xi_3$ remains unaltered under the symmetric group on the $x_i$ and is rationally expressible in terms of the $a_i$. A computation gives

$$\xi_1\xi_2\xi_3 = -\frac{32}{a_0^3}(a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3),$$

and this is the restriction imposed upon our choice of sign for $\xi_i$. If in the last calculation we obtain a root $x_i$ that does not satisfy the biquadratic, we have only to change the sign of any one $\xi_i$ to correct our mistake—if we did not blunder before we came to the $\xi_i$.

Example:

$$x^4 - 2x^3 + 3x^2 + 2x - 4 = 0$$

$$a_0 = 1,\ a_1 = -\tfrac{1}{2},\ a_2 = \tfrac{1}{2},\ a_3 = \tfrac{1}{2},\ a_4 = -4.$$

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2 = -\tfrac{9}{4}$$

$$g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} = -1\tfrac{3}{8}$$

$$\varphi_1 = \frac{2}{a_0}\left(-27g_3 - 3\sqrt{81g_3^2 - 3g_2^3}\right)^{1/3} = -3$$

$$\varphi_2 = \frac{2}{a_0}\left(-27g_3 + 3\sqrt{81g_3^2 - 3g_2^3}\right)^{1/3} = 9$$

$$y_1 = \frac{2a_2}{a_0} + \frac{1}{3}(\varphi_1 + \varphi_2) = 3$$

$$y_2 = \frac{2a_2}{a_0} + \frac{1}{3}(\omega^2\varphi_1 + \omega\varphi_2) = 2\sqrt{-3}$$

$$y_3 = \frac{2a_2}{a_0} + \frac{1}{3}(\omega\varphi_1 + \omega^2\varphi_2) = -2\sqrt{-3}$$

$$\xi_1 = \frac{2}{a_0}\sqrt{4a_1^2 - 6a_0a_2 + a_0^2y_1} = -2$$

$$\xi_2 = \frac{2}{a_0}\sqrt{4a_1^2 - 6a_0a_2 + a_0^2y_2}$$

$$= 2\sqrt{-2 + 2\sqrt{-3}} = -2 - 2\sqrt{-3}$$

$$\xi_3 = \frac{2}{a_0}\sqrt{4a_1^2 - 6a_0a_2 + a_0^2y_3}$$

$$= 2\sqrt{-2 - 2\sqrt{-3}} = -2 + 2\sqrt{-3}$$

such that

$$\xi_1\xi_2\xi_3 = -\frac{32}{a_0^3}(a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3) = -32$$

$$\begin{aligned} \xi_1 &= x_1 + x_2 - x_3 - x_4 = -2 \\ \xi_2 &= x_1 - x_2 + x_3 - x_4 = -2 - 2\sqrt{-3} \\ \xi_3 &= x_1 - x_2 - x_3 + x_4 = -2 + 2\sqrt{-3} \\ & \quad x_1 + x_2 + x_3 + x_4 = 2 \end{aligned}$$

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$$\begin{aligned} x_1 &= -1 \\ x_2 &= 1 \\ x_3 &= 1 - \sqrt{-3} \\ x_4 &= 1 + \sqrt{-3}. \end{aligned}$$

## CHAPTER IX

## MORE ABOUT GROUPS

### §49. ISOMORPHIC GROUPS

To continue the study of equations, we have to know more about groups and something about domains.

Two groups may be subordinated as group and subgroup, or they may be coördinated by a common property. Coördinated in such a manner that their permutations match under a common law of combination, they are called **isomorphic** and conveniently denoted by $G$ and $\Gamma$.

If the permutations of two groups

$$G = 1, s_2, s_3, \ . \ . \ . \ , s_r$$
$$\Gamma = 1, \sigma_2, \sigma_3, \ . \ . \ . \ , \sigma_r$$

correspond one to one in such a way that

$$\sigma_i \sigma_j = \sigma_k$$

when

$$s_i s_j = s_k,$$

the groups are said to be **simply isomorphic.** Two groups simply isomorphic with a third evidently are simply isomorphic with one another.

Since the product of two transformed permutations is the transformed product of those permutations:

$$t^{-1}s_i t \, . \, t^{-1}s_j t = t^{-1}s_i s_j t,$$

the permutations of every group are in a one to one correspondence with their transforms in the transformed group:

$$\begin{aligned} G &= \ . \ . \ . \ , \quad s_i, \quad\quad s_j \ , \ . \ . \ . \ , \quad\quad s_i s_j, \ . \ . \ . \\ t^{-1}Gt &= \ . \ . \ . \ , t^{-1}s_i t, t^{-1}s_j t, \ . \ . \ . \ , t^{-1}s_i s_j t, \ . \ . \ . \end{aligned}$$

Hence

(57) **every group is simply isomorphic with its transform,** and conjugate groups are simply isomorphic with each other.

If the group $G$ is isomorphic with the group $\Gamma$ in such a way that to the identical permutation in $\Gamma$ corresponds not one permutation but a set

$$s_1, \ldots, s_n$$

of them in $G$, this set of permutations is a group since the product

$$s_i\, s_j = s_k$$

of any two such permutations corresponds to

$$1 \,.\, 1 = 1$$

in $\Gamma$ and is in the set by its very correspondence to identity in $\Gamma$.

The transform of this group

$$\{s_i\} = s_1, \ldots, s_n$$

by any permutation $t$ of $G$ corresponding to the permutation $\tau$ of $\Gamma$ contains only such permutations of $G$ as correspond to the transform

$$\tau^{-1} \,.\, 1 \,.\, \tau = 1$$

in $\Gamma$ and therefore is the group itself:

$$t^{-1}\{s_i\}t = \{s_i\}.$$

It appears that this group is a normal subgroup of $G$:

$$\{s_i\}_1^n = N.$$

If the permutation $t_2$ of $G$ corresponds to the permutation $\tau_2$ of $\Gamma$, all the permutations $Nt_2$ do so as they correspond to

$$1 \,.\, \tau_2 = \tau_2$$

in $\Gamma$, and no other permutations will.[1] It follows that to every permutation in $\Gamma$ corresponds an equal number of permutations in $G$. If that number is $n$, we say that $G$ and $\Gamma$ are **$n$ to 1 isomorphic,** which we denote by $(n, 1)$-isomorphic. Hence we conclude:

(58) **Whenever two groups $G$ and $\Gamma$ are $(n, 1)$-isomorphic, we have in correspondence:**

$$\boxed{\begin{array}{l} G = N + Nt_2 + \ldots + Nt_\rho \\ \Gamma = 1, \quad \tau_2, \quad \ldots, \quad \tau_\rho \end{array}},$$

**where $N$ is a normal subgroup of $G$**

and $\rho$ is the order of $\Gamma$ or index of $N$ in $G$; the order of $N$ is

$$n = \frac{r}{\rho},$$

$r$ being the order of $G$.

[1] Compare §17.

The permutations on the $x_i$ contained in a partition of $G$ with respect to $N$ are equivalent in their effect upon conjugate functions of the $x_i$ that belong to $N$ or to groups including $N$ as greatest common subgroup.[1] Hence the group $G$ on the $x_i$ is $(n, 1)$-isomorphic with a group $\Gamma$ of permutations between those functions. It follows from proposition (49) that with $N$ also $\Gamma$ exists.

An example of such isomorphism occurred in §33 and was tabulated there.

Two groups $G$ and $G'$ may be isomorphic in such a manner that to a normal subgroup of $G$ corresponds a normal subgroup of $G'$. If they are, the respective groups $\Gamma$ and $\Gamma'$ are simply isomorphic.

## §50. TRANSITIVE GROUP

'he letters $x_1$ and $x_i$ are said to be connected by a group if there is in the group a permutation that replaces $x_1$ by $x_i$, this then implying that there is in the group the inverse permutation replacing $x_i$ by $x_1$.

If a group connects two letters $x_i$ and $x_k$ with $x_1$, it connects them with one another; for if it contains a permutation replacing $x_i$ by $x_1$ and a permutation replacing $x_1$ by $x_k$, then it contains also their product which replaces $x_i$ by $x_k$.

A group connecting $x_1$ with every one of the other letters is called **transitive.** As it connects all letters with $x_1$, it connects every letter with every other letter. If it does not do so, it is called **intransitive.** For instance:[2]

$$V = 1, (12)(34), (13)(24), (14)(23)$$

is transitive because its permutations replace 1 by 2 and 3 and 4;

$$W = 1, (12), (34), (12)(34)$$

is intransitive because its permutations replace 1 by 2, but do not replace 1 by either 3 or 4.

(59) **A cyclic group $\{s\}$ is transitive whenever the permutation $s$ is circular,**

as in

$$\{s\} = 1, (1234), (13)(24), (1432)$$

[1] This is true only for a normal subgroup $N$, by §§32 and 33.

[2] These groups appeared in §31.

with

$$s = (1234)$$

or

$$s = (1432).$$

This is to say that every circular group is transitive.

By the intransitive group $W$ the letters $x_1$ and $x_2$ are connected, and so are the letters $x_3$ and $x_4$. Thus the four letters $x_i$ of the group are divided by the group into two sets in such a way that each letter is connected with those of its set, but is not connected with those of the other set. Such sets into which the letters of an intransitive group divide are called **intransitive systems**[1] of the group, and the number of letters in a system is its **degree.**

If two transitive groups operate on distinct letters, the group that we obtain by multiplying in all possible ways their permutations is an intransitive group. Thus the intransitive group $W$ is obtained by multiplying the permutations of two transitive groups

$$H = 1, (12)$$

and

$$H' = 1, (34).$$

But we cannot say conversely that every intransitive group is obtained by multiplying the permutations of transitive groups, for

$$G = 1, (12)(34)$$

is not.

While the order of every group of degree $n$ is a divisor of $n!$ by proposition (23), we now prove that

(60) **the order of a transitive group of degree $n$ is a multiple of $n$.**

Let $G$ be a transitive group of degree $n$ such that its permutation $t_i$ replaces $x_1$ by $x_i$, which is to say that

$$t_1 = 1$$

while

$t_2$ replaces $x_1$ by $x_2$
$t_3$ replaces $x_1$ by $x_3$
. . . . . . .
$t_n$ replaces $x_1$ by $x_n$.

[1] Also called transitive systems or sets.

Those permutations of $G$ that do not displace $x_1$—of which there is at least the identical permutation—form a subgroup $H$ of $G$ because the product of two such permutations, not displacing $x_1$ either, must be in $H$.

If now $t_2$ replaces $x_1$ by $x_2$, all permutations in $Ht_2$ evidently do the same, and no other permutations will; all permutations in $Ht_3$ replace $x_1$ by $x_3$, and so on according to the table:

$$\begin{array}{lll} H = 1, s_2, \quad \ldots, s_r, & & x_1 \to x_1 \\ Ht_2 = t_2, s_2t_2, \quad \ldots, s_rt_2, & & x_1 \to x_2 \\ \quad\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & & \\ Ht_n = t_n, s_2t_n, \quad \ldots, s_rt_n, & & x_1 \to x_n. \end{array}$$

This table contains all the permutations of $G$, since no permutation of $G$ can help leaving $x_1$ unaltered or replacing it by some $x_i$. Denoting the order of $G$ by $r_g$ and that of $H$ by $r_h$, we therefore have

$$r_g = nr_h.$$

If identity alone leaves $x_1$ unaltered, we have

$$r_h = 1$$

and

$$r_g = n,$$

as for the group $V$.

A transitive group of degree and order $n$ is called **regular,** because its permutations are all regular. For instance is $V$ a regular group; other examples are the two groups $C$ in §51, while the group $W$ is of degree and order four but not regular.

If all permutations of $H$ leave $x_1$ unaltered, all permutations of

$$H_i = t_i^{-1}Ht_i$$

leave $x_i$ unaltered since the permutation $t_i^{-1}$ replaces $x_i$ by $x_1$, the permutations of $H$ do not displace $x_1$, and the permutation $t_i$ replaces $x_1$ by $x_i$. Other permutations of $G$ displace $x_i$, and it appears that

(61) **the permutations of $G$ which leave any one letter $x_i$ unaltered compose conjugate subgroups of $G$.**

This is self-evident if we recall the rule of transforms, and we infer that

(62) **any subgroup $H_i$ is transitive in the letters it acts upon if the subgroup $H$ is so.**

A transitive group can be **simply transitive,** as the group $V$ is. It is **doubly transitive** if it connects every pair of letters with every other pair; it can be multiply transitive and, when symmetric of degree $n$, obviously is $n$-fold transitive.

Permutations of a doubly transitive group leaving pairs of letters $x_i$ unaltered compose conjugate subgroups, as those do leaving single letters $x_i$ unaltered.

For a group $G$ to be doubly transitive, it is necessary that its subgroup $H$ whose permutations leave one letter $x_1$ unaltered should be transitive itself in another letter $x_i$, because the subgroup $H$ must contain a permutation replacing any

$$x_1x_i \longrightarrow x_1x_k.$$

And this condition is sufficient since in that subgroup conjugate with $H$ whose permutations leave $x_k$ unaltered there is a permutation replacing

$$x_1x_k \longrightarrow x_jx_k,$$

and hence there is in $G$ a permutation replacing

$$x_1x_i \longrightarrow x_jx_k.$$

But connecting every pair of letters with $x_1x_i$, the group $G$ connects every pair of letters with every other pair, and we conclude that

(63) **a group is doubly transitive if its subgroup leaving one letter unaltered is transitive in another.**

The order of the subgroup $H$ is by proposition (60)

$$r_h = (n - 1)r_k,$$

where $r_k$ is the order of a subgroup of $H$ leaving one of the $n - 1$ letters acted upon by $H$ unaltered. Hence the order of a doubly transitive group $G$ is a multiple of $n$ and $n - 1$:

$$r_g = n(n - 1)r_k.$$

## §51. IMPRIMITIVE GROUP

Transitive groups may be primitive or imprimitive. If the letters operated on by a transitive group divide into equal sets in such a way that the permutations of the group either interchange the letters within the sets or replace them by the letters of other sets but never break up the sets, then the group is called

**imprimitive or non-primitive,** and the sets are called **imprimitive systems** of the group. Otherwise the group is called **primitive.**

For instance, the transitive group

$$C = 1, (1234), (13)(24), (1432)$$

on four letters $x_i$ is imprimitive because it has the imprimitive systems

$$x_1, x_3 \mid x_2, x_4;$$

the transitive group

$$V = 1, (12)(34), (13)(24), (14)(23)$$

is primitive because it has no such systems.

Those permutations of an imprimitive group $G$ that do not displace the imprimitive systems form a subgroup of $G$, since the product of any two such permutations does not displace the imprimitive systems either. This subgroup of permutations only interchanging the letters within the imprimitive systems is normal in $G$ by proposition (58), so that we may denote it by $N$ and set

$$G = N + Nt_2 + \ . \ . \ . \ + Nt_\rho.$$

For the permutations of this subgroup correspond with identity in the group

$$\Gamma = 1, \tau_2, \ . \ . \ . \ , \tau_\rho$$

composed of permutations between the imprimitive systems themselves, as caused by the permutations of $G$ between the $x_i$ and obviously isomorphic with $G$. In the example given above we have

$$N = 1, (13)(24).$$

Since the normal subgroup $N$ operates on letters of different imprimitive systems but does not connect them it is intransitive and we have the proposition:

(64) **The permutations of an imprimitive group not displacing its imprimitive systems form a normal subgroup which is intransitive.**

Conversely,

(65) **if a transitive group has a normal subgroup which is intransitive, then the group is imprimitive.**

Suppose that $t$ is a permutation in the transitive group $G$ connecting two letters in different intransitive systems of the

normal subgroup $J$ of $G$. The transform $t^{-1}Jt$ is obtained by proposition (29) if we operate the permutation $t$ within the cycles of $J$. But

$$t^{-1}Jt = J$$

and consequently has the same intransitive systems. Hence the permutation $t$, having replaced in the cycles of $J$ a letter of one intransitive system by a letter of another, must have so replaced every letter interchanging the two systems. This means that the permutations of $G$ either leave the intransitive systems of $J$ unaltered, as the permutations of $J$ do, or interchange them, as the permutation $t$ does, whence it appears that $G$ is imprimitive as our proposition states.

Since $N$ contains all permutations which do not displace the imprimitive systems of $G$, it follows that $J$ is a subgroup of $N$ if it is not identical with it.

It is clear that the intransitive systems of $J$ or $N$ are also the imprimitive systems of $G$.

No imprimitive group can be more than simply transitive, for its permutations can replace a pair of letters which is contained in an imprimitive system not by any other pair of letters, but by such pairs alone as belong to some imprimitive system.

If the letters $x_i$ operated on by the permutations of a group divide into $k$ imprimitive systems with $m$ letters each, the letters of any one system can interchange in $m!$ ways, which gives for all systems $(m!)^k$ combinations; and the systems can interchange in $k!$ ways. Hence the greatest number of permutations possible for imprimitive groups with such systems is

$$r = (m!)^k \,.\, k!$$

In case of four letters $x_i$ the greatest imprimitive group with the imprimitive systems

$$x_1, x_2 \,|\, x_3, x_4$$

has the order

$$r = 8$$

and is the group of the function

$$\psi = x_1x_2 + x_3x_4$$

given in §17.

An imprimitive group may be imprimitive in more than one way. For instance, the transitive group
$C = 1, (123456), (135)(246), (14)(25)(36), (153)(264), (165432)$
on six letters $x_i$ has the imprimitive systems

$$x_1, x_3, x_5 \mid x_2, x_4, x_6$$

with the normal and intransitive subgroup

$$N = 1, (135)(246), (153)(264);$$

and also the imprimitive systems

$$x_1, x_4 \mid x_2, x_5 \mid x_3, x_6$$

with the normal and intransitive subgroup

$$N' = 1, (14)(25)(36).$$

## §52. QUOTIENT-GROUP

If we multiply two groups $H$ and $H'$, which is to say multiply in all possible ways their permutations, we obtain a definite group

$$G = \{H, H'\}.$$

We call $G$ the **product** of $H$ and $H'$, and both $H$ and $H'$ are contained in $G$ as subgroups.

But the order of $G$ is not in general the product of the orders of $H$ and $H'$. For instance: if

$$H = 1, (12)$$

and

$$H' = 1, (34),$$

then

$$G = 1, (12), (34), (12)(34);$$

if

$$H = 1, (12)$$

and

$$H' = 1, (12)(34),$$

then

$$G = 1, (12), (12)(34), (34),$$

the order being equal to the product in both cases. But if

$$H = 1, (12)$$

and

$$H' = 1, (13),$$

we have[1]

$$G = 1, (12), (13), (23), (123), (132)$$

with an order greater than the product, while we have

$$G = \{H, H'\} = H$$

if $H'$ is a subgroup of $H$, the order now being less than the product.

The product of two groups is called **direct** if every permutation of one group is commutative with every permutation of the other and the two groups have no permutation but identity in common. It is readily seen that

(66) **the order of the direct product of two groups is the product of their orders.**

For instance, the product of

$$H = 1, (12)$$

and

$$H' = 1, (34)$$

is direct.

Pursuing the analogy with numbers, we may well inquire whether division of groups has a meaning.

If we arrange the permutations of

$$G = H + Ht_2 + \dots + Ht_r$$

into partitions with respect to $H$, the permutations

$$1, t_2, \dots, t_r$$

of $H'$ do not necessarily form a group. Indeed, it may not even be possible to pick them out so that they do form a group. And again, it may be possible to pick out more than one group $H'$. Thus for

$$G = \begin{cases} 1, \quad (12), \quad (34), \quad (12)(34) = H \\ (13)(24), (1423), (1324), (14)(23) = Ht_2, \end{cases}$$

we find

$$H' = 1, (13)(24)$$

or

$$H' = 1, (14)(23).$$

All then we can say is that in every partition $Ht_i$ of a group $G$ with respect to its subgroup $H$ there may be contained a permutation $q_i$ such that we can set

$$G = H + Hq_2 + \dots + Hq_\rho$$

[1] Symmetric group by proposition (33).

and that the permutations $q_i$ compose a group. This group we call the **quotient** of $G$ divided by $H$ and denote by $Q$:

$$Q = 1, q_2, \ . \ . \ . \ , q_\rho.$$

The quotient is in general neither certain nor unique, but evidently it is so when $G$ is the direct product of $H$ and $H'$, for then

$$Q = H'.$$

Here the matter ends unless the subgroup is normal. In that case the quotient is not certain either; and it is not any more definite as we see from the example:

$$G = \begin{cases} 1, (12)(34), (13)(24), (14)(23) = N \\ (12), (34), \quad (1324), \quad (1423) \ \ = Nt_2, \end{cases}$$

where

$$Q = 1, (12)$$

or

$$Q = 1, (34).$$

But if the quotient of

$$G = N + Nt_2 + \ . \ . \ . \ + Nt_\rho$$

divided by $N$, say

$$Q = 1, q_2, \ . \ . \ . \ , q_\rho,$$

exists, it is isomorphic with $G$ in such a manner that its identical permutation corresponds to $N$ in $G$, and it is therefore simply isomorphic with an always existing[1] and readily constructed group

$$\Gamma = 1, \tau_2, \ . \ . \ . \ , \tau_\rho$$

of permutations between conjugate functions belonging to $N$. This gives a good deal of information about $Q$ since simply isomorphic groups obey the same law of combination.

If we emancipate our notion of groups from its connection with permutations and regard a group as exhibiting on elements, whether permutations or otherwise, a definite law of combination for which alone it stands, we obtain the concept of an **abstract group.**[2]

Abstractly speaking, simply isomorphic groups are identical; and the quotient $Q$, together with the group $\Gamma$, merges into an abstract group called the **abstract quotient** of $G$ divided by $N$, or

[1] Cf. §49, near end.

[2] Cf. the note at the end of this chapter.

**factor-group,** and denoted by $G/N$. We take it as composed of the elements $Nt_i$; they contain the product of any two elements among them, since by proposition (32)

$$Nt_i \,.\, Nt_j = NNt_it_j = Nt_k,$$

and contain $N$ as the identical element:

$$\begin{aligned} G/N &= N, Nt_2, \ldots, Nt_\rho \\ Q &= 1, \quad q_2, \ldots, \quad q_\rho \\ \Gamma &= 1, \quad \tau_2, \ldots, \quad \tau_\rho. \end{aligned}$$

With the group $\Gamma$, the factor-group $G/N$ always exists, and we can note:

(67) **If a group $G$ has a normal subgroup $N$, there exists the factor-group $G/N$ abstractly identical with the group $\Gamma$ of permutations operated by $G$ on functions belonging to $N$ or to groups containing $N$ as greatest common subgroup. The order of $G/N$ is equal to the index of $N$ in $G$.**

By proposition (25) it follows that $G/N$ is cyclic if the index of $N$ in $G$ is prime, and this obviously applies also to $Q$ and $\Gamma$.

### §53. SUBGROUPS OF QUOTIENT-GROUP

To every subgroup of the quotient $Q$ that we obtained dividing $G$ by $N$ corresponds isomorphically a subgroup of $G$ containing $N$, and a similar relation holds true for $\Gamma$.

To verify this, let a subgroup of $Q$ be

$$R = 1, q_\alpha, q_\beta, \ldots$$

Then the partitions of $G$ with respect to $N$ which correspond to the permutations of $R$ evidently compose a subgroup

$$H = N + Nt_\alpha + Nt_\beta + \ldots$$

of $G$ containing $N$. But $R$ is isomorphic with $H$ in such a manner that its identical permutation corresponds to $N$ in $H$, whence its law of combination is given by the factor-group

$$H/N = N, Nt_\alpha, Nt_\beta, \ldots$$

If $R$ is normal in $Q$, also $H$ is normal in $G$, since the transform of $Nt_\alpha$ by a permutation of $G$ corresponds to the transform of $q_\alpha$ by a permutation of $Q$, and with this transform in $R$ the other is in

$H$. In this case $G$ is isomorphic with $Q$ in such a manner that the permutations of $H$ in $G$ correspond to the permutations of $R$ in $Q$:

$$G = H + Ht_a + Ht_b + \ldots$$
$$Q = R + Rq_a + Rq_b + \ldots,$$

for with[1]

$$Ht_a \,.\, Ht_b = Ht_at_b$$

also

$$Rq_a \,.\, Rq_b = Rq_aq_b,$$

and

$$q_aq_b = q_c$$

corresponds to

$$t_at_b = t_c.$$

It follows that

$$Q/R = G/H,$$

and the index of $R$ in $Q$ equals the index of $H$ in $G$.

Replacing $Q$ and $R$ by abstract groups, we may write

$$\{G/N\}/\{H/N\} = \{G/N\},$$

which agrees with the rules as to division of elementary algebra, and we have the proposition:

(68) **To every subgroup of $G/N$ corresponds a subgroup of $G$ containing $N$; if one is normal, the other is also.**

## §54. MAXIMUM NORMAL SUBGROUP

A normal subgroup of $G$ contained in no other normal subgroup of $G$ is called a **maximum normal subgroup** of $G$ and may be denoted by $\bar{N}$.

This does not imply that there is in $G$ no normal subgroup of greater or the same order, only that it does not contain a maximum normal subgroup. The group

$G = 1, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423),$

for instance, has two maximum normal subgroups:

$$\bar{N} = 1, (12)(34), (13)(24), (14)(23)$$
$$\bar{N}' = 1, (12), (34), (12)(34).$$

To maximum normal subgroups applies the proposition:

[1] Cf. proposition (32).

(69) **If $\bar{N}$ and $\bar{N}'$ are maximum normal subgroups of $G$ and $D$ is their greatest common subgroup, then $D$ is a maximum normal subgroup of both $\bar{N}$ and $\bar{N}'$ for which**

$$G/\bar{N} = \bar{N}'/D$$

**and**

$$G/\bar{N}' = \bar{N}/D.$$

To prove this, we notice that $\bar{N}\bar{N}'$ is a normal subgroup of $G$, since any permutation $t$ of $G$ gives

$$t^{-1}\bar{N}\bar{N}'t = t^{-1}\bar{N}t \,.\, t^{-1}\bar{N}'t = \bar{N}\bar{N}';$$

and since $\bar{N}\bar{N}'$ contains the maximum normal subgroups $\bar{N}$ and $\bar{N}'$, it even is identical with $G$:

$$G = \bar{N}\bar{N}'.$$

Setting

$$\bar{N}' = D + Dt_2 + \ . \ . \ . + Dt_\rho,$$

we have

$$G = \bar{N}\bar{N}' = \bar{N}(D + Dt_2 + \ . \ . \ . + Dt_\rho)$$

or

$$G = \bar{N} + \bar{N}t_2 + \ . \ . \ . + \bar{N}t_\rho,$$

because $D$ is a subgroup of $\bar{N}$ and therefore

$$\bar{N}D = \bar{N}.$$

The partitions of $G$ with respect to $\bar{N}$ are all distinct as from

$$\bar{N}t_i = \bar{N}t_k$$

would follow

$$\bar{N}t_it_k^{-1} = \bar{N},$$

showing that $t_it_k^{-1}$ is a permutation of $\bar{N}$. Since $t_it_k^{-1}$ also is in $\bar{N}'$ which contains any combination of the $t_i$, it would have to be in $D$. But

$$Dt_it_k^{-1} = D$$

gives

$$Dt_i = Dt_k$$

which is untrue.

Now with

$$\bar{N}t_i \,.\, \bar{N}t_j = \bar{N}t_it_j = \bar{N}t_k$$

also

$$Dt_i \,.\, Dt_j = Dt_it_j = Dt_k,$$

for $D$ is normal by proposition (39),[1] and it follows that the groups

$$G = \bar{N} + \bar{N}t_2 + \ . \ . \ . + \bar{N}t$$

[1] The proof of proposition (39) given for conjugate subgroups remains true for maximum normal subgroups.

and

$$\bar{N}' = D + Dt_2 + \ldots + Dt_p$$

are isomorphic in such a manner that the permutations of $D$ in $N'$ correspond with the permutations of $\bar{N}$ in $G$, whence

$$G/\bar{N} = \bar{N}'/D.$$

Interchanging $N$ and $\bar{N}'$, we prove likewise that

$$G/\bar{N}' = \bar{N}/D.$$

All these factor-groups are simple since by proposition (68) it appears that only when $G/\bar{N}$ is simple has $G$ no normal subgroup containing $\bar{N}$ and is $\bar{N}$ a maximum normal subgroup of $G$.

Again, since these factor-groups are simple is $D$ a maximum normal subgroup of both $\bar{N}$ and $\bar{N}'$.

## §55. CONSTANCY OF COMPOSITION-FACTORS

The proposition we proved leads to an important theorem due to **Jordan and Hölder:**

(70) **If a group has more than one composition-series, their factor-groups are identical except for the sequence.**

Suppose the proposition is true for groups whose order is the product of fewer than $n$ prime numbers, and let $G$ be a group whose order is the product of $n$ prime numbers. If we can prove that the proposition is true for $G$, it will by induction be true in general, since the constancy of factor-groups is evident for groups of prime order, which are simple.

Let two composition-series of $G$ be

$$G \quad N \quad I \quad \ldots \quad 1$$

and

$$G \quad N' \quad I' \quad \ldots \quad 1,$$

made up of maximum normal subgroups by the definition of a composition-series.

Let the greatest common subgroup of $N$ and $N'$ be $D$; then we can construct by proposition (69) two other composition-series of $G$ containing $D$ and identical from $D$ on:

$$G \quad N \quad D \quad J \quad \ldots \quad 1$$
$$G \quad N' \quad D \quad J \quad \ldots \quad 1,$$

having the factor-groups

$$G/N \quad N/D \quad D/J \quad \ldots$$
$$G/N' \quad N'/D \quad D/J \quad \ldots$$

But

$$G/N = N'/D$$

and

$$G/N' = N/D$$

by proposition (69), and it appears that the last two composition-series have identical sets of factor-groups with the sequence of the first two factor-groups inverted.

Again, the two composition-series

$$N \quad I \quad \ldots \quad 1$$

and

$$N \quad D \quad \ldots \quad 1$$

as well as the two composition-series

$$N' \quad I' \quad \ldots \quad 1$$

and

$$N' \quad D \quad \ldots \quad 1$$

have identical factor-groups by assumption, since the order of $G$ is the product of $n$ primes and consequently the order of both $N$ and $N'$ the product of fewer than $n$ primes.

It follows that with the two composition-series

$$G \quad N \quad D \quad \ldots \quad 1$$

and

$$G \quad N' \quad D \quad \ldots \quad 1$$

also the two composition-series

$$G \quad N \quad I \quad \ldots \quad 1$$

and

$$G \quad N' \quad I' \quad \ldots \quad 1$$

have identical factor-groups, which proves the theorem.

The existence for the symmetric group of a composition-series with prime composition-factors is a sufficient condition for the solvability of the general equation, as we know. The theorem of Jordan-Hölder relieves us of the necessity to investigate all possible composition-series, since they all have the same composition-factors.

In a sense which is obvious for the symmetric group, and which will become so for other groups in connection with the theory of

Galois, we call a group whose factors of composition are all prime a **soluble group.**[1] Hence we conclude that

(71) **all factor-groups in the composition-series of a soluble group are cyclic;**

this is true by proposition (25), for the order of any such factor-group is prime by proposition (67).

## §56. ABELIAN GROUP

Just as the transform of a normal subgroup of the group $G$ by any permutation of $G$ is the same subgroup, so it may happen that the transform of a permutation $z$ of $G$ by any permutation $t$ of $G$ is the same permutation $z$. Such a permutation is called **normal** in $G$.

If $z'$ is another normal permutation of $G$, then also the product $zz'$ is normal in $G$, for

$$t^{-1}zz't = t^{-1}zt \,.\, t^{-1}z't = zz'.$$

Hence all the normal permutations of $G$ form a subgroup, and this subgroup, normal not only as a whole but in every permutation, is called the **central subgroup**[2] of $G$ and may be denoted by $Z$. It is needless to say that every subgroup of $Z$ is normal in $G$. A trivial case of a central subgroup presents itself in identity.

Being normal in $G$,

(72) **the permutations of the central subgroup of $G$, and such permutations alone, are commutative with every permutation of $G$,**

since from

$$t^{-1}zt = z$$

follows that

$$zt = tz,$$

and conversely.

If the permutations of a group are all commutative, we call the group **commutative or Abelian,** in memory of a man who excelled in genius and misfortune.[3] It is clear that subgroups of an Abelian

[1] A soluble group is also called a **metacyclic group,** and a solvable equation a **metacyclic equation.** This has to be remembered well if one is to understand the literature, but we shall use the term in another sense.

[2] Called "Zentrum" in German.

[3] Abel lived 1802–1829.

group also are Abelian. Denoting an Abelian group by $A$ or $\langle A \rangle$, we observe that

(73) **the central subgroup is Abelian:**

$$\boxed{\{Z\} = \langle A \rangle}.$$

Every subgroup of an Abelian group, indeed every permutation, is normal since

$$t^{-1}st = t^{-1}ts = s$$

for any permutations $s$ and $t$ of the group. Therefore we note that

(74) **an Abelian group is its own central subgroup.**

A simple case of an Abelian group is a cyclic group: the cyclic group

$$\{s\} = s, s^2, \ldots s^r \qquad [s^r = 1$$

is Abelian since

$$s^i s^j = s^j s^i = s^{i+j}.$$

It is quickly verified that

(75) **every cyclic group of order $r$ has a cyclic subgroup, and only one, of every order which is a divisor of $r$.**

In particular, if $r$ has a prime factor $p$ and

$$kp = r,$$

the cyclic group $\{s\}$ has the cyclic subgroup

$$\{s^p\} = s^p, s^{2p}, \ldots, s^{kp} \qquad [s^{kp} = 1$$

of order $k$ and index $p$. Hence it follows that

(76) **every cyclic group is soluble.**

In general it is true that

(77) **an Abelian group whose order is divisible by a prime number $p$ contains a permutation of order $p$.**

For suppose the Abelian group $A$ is generated by the permutations

$$s_1, s_2, s_3, \ldots$$

and let $r_1$ be the order of $s_1$. If then $r_2$ is the order of $s_2$, the permutation $s_2^{r_2}$ obviously appears in $\{s_1\}$ because

$$s_2^{r_2} = 1.$$

But it may happen that already a lower power of $s_2$ appears in $\{s_1\}$, determining what is called the **order of $s_2$ relative to $\{s_1\}$.**

Let it be $\rho_2$. The order $\rho_2$ of $s_2$ relative to $\{s_1\}$ is a divisor of the **absolute order** $r_2$ of $s_2$, for all the permutations

$$s_2^{\rho_2}, s^{2\rho_2}, s^{3\rho_2}, \ldots$$

and these alone,[1] are in $\{s_1\}$; but so is $s_2^{r_2}$, which therefore is one of them, and

$$r_2 = l\rho_2.$$

Similarly, let $\rho_3$ be the order of $s_3$ relative to $\{s_1, s_2\}$ while $r_3$ is the absolute order of $s_3$, and so on. The order $r$ of $A$ then is

$$r = r_1\rho_2\rho_3 \ldots,$$

and all permutations of $A$ can be expressed in the form

$$s_1^{i_1}s_2^{i_2}s_3^{i_3} \ldots$$

where

$$\begin{aligned} i_1 &= 1, 2, \ldots, r_1 \\ i_j &= 1, 2, \ldots, \rho_j \\ j &= 2, 3, \ldots \end{aligned}$$

If now the order $r$ of $A$ is divisible by a prime number $p$, it follows that divisible by $p$ is also $r_1$ or some $\rho_\iota$ which itself divides $r_\iota$. For some $r_\iota$ we thus have

$$r_i = kp,$$

and $s_\iota^k$ is a permutation in $A$ of order $p$ since

$$(s_i^k)^p = 1.$$

This enables us to prove that

(78) **an Abelian group of order $r$ contains a subgroup of any order $h$ which is a divisor of $r$.**

The proposition is true for any order whose factors are all prime, since corresponding to any such factor there is by proposition (77) a permutation in the Abelian group $A$ whose powers compose such a subgroup.

We assume that the proposition is true for any order which is smaller than $r$; if we can prove that it is true also for $r$, it is by induction true in general.

Let $p$ be a prime factor of $h$ and hence of $r$, and let a permutation of order $p$ in $A$ be $s$. The quotient-group $A/\{s\}$ is Abelian of order[2] $r/p$, which is smaller than $r$, and by assumption it

[1] Since otherwise a power of $s_2$ lower than $\rho_2$ would be in $\{s_1\}$.

[2] Cf. proposition (67).

contains a subgroup of order $h/p$. But to this subgroup of $A/\{s\}$ corresponds in $A$ by proposition (68) a subgroup of index

$$\frac{r}{p} \div \frac{h}{p} = \frac{r}{h}$$

and hence of order $h$, which proves our proposition.

It follows that not only every cyclic group, but

(79) **every Abelian group is soluble.**

If we write the order of $A$ as

$$r = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k},$$

where every $p_i$ is prime, it appears from proposition (78) that $A$ has a subgroup $H_i$ of every order $p_i^{\alpha_i}$, and only one. For if it had two, their product would be a subgroup of order $p_i^{\beta_i}$ with $\beta_i > \alpha_i$, and $p_i^{\beta_i}$ would have to divide $r$ which is impossible.

The different subgroups $H_i$ have no permutation but identity in common, since any subgroup $H_i$ contains only permutations whose order is a power of $p_i$, so as to divide the order $p_i^{\alpha_i}$ of $H_i$. The direct product of the $H_i$ has the order

$$p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} = r$$

by proposition (66), and having the same order as $A$ it is identical with $A$. Hence we conclude that

(80) **every Abelian group is the direct product of Abelian subgroups whose orders are powers of prime numbers.**

If

$$H_1 = \{s_1\},\ H_2 = \{s_2\},\ \ldots$$

and

$$s = s_1 s_2 \ldots,$$

then

$$A = \{s\}.$$

Therefore $A$ is cyclic if the $H_i$ are so.

## §57. THEOREM OF CAUCHY

Permutations which are conjugate under a group $G$ make up what we call a **class of permutations** in $G$. It appears that

(81) **a group is composed of classes which have no common permutations; the permutations of a class have the same order,**

since conjugate permutations are similar by proposition (30).

(82) **Permutations of a group $G$ which are commutative with a given permutation $c$ of $G$ form a subgroup of $G$ whose index equals the number of permutations in the class of $c$.**

They form a subgroup $H$ because with the permutations $s_1$ and $s_2$ contained in $H$ also their product $s_1s_2$ qualifies for $H$, since

$$s_1s_2 \,.\, c = s_1cs_2 = c \,.\, s_1s_2.$$

If two permutations $t_1$ and $t_2$ which are in $G$ but not in $H$ transform $c$ into the same conjugate permutation, then

$$t_1^{-1}ct_1 = t_2^{-1}ct_2$$

or

$$c \,.\, t_1t_2^{-1} = t_1t_2^{-1} \,.\, c.$$

Therefore the permutation $t_1t_2^{-1}$ is in $H$, and from

$$Ht_1t_2^{-1} = H$$

or

$$Ht_1 = Ht_2$$

we infer that $G$ has as many partitions with respect to $H$ as there are permutations conjugate to $c$ in $G$. But this is to say that the index of $H$ in $G$ equals the number of permutations in the class of $c$.

It follows that the number of permutations in the class of $c$ divides the order of $G$, since the index of $H$ does. We may note that the subgroup $H$ of $G$ is called the **normalizer** of $c$ in $G$.

If every permutation of $G$ is commutative with the permutation $c$, the number of permutations in the class of $c$ is one, which is to say that $c$ is a normal permutation as we should expect it to be by proposition (72). A trivial case presents itself when the permutation $c$ is identity.

Now we can prove the **Theorem of Cauchy**:[1]

(83) **If the order of a group is divisible by a prime number $p$, the group contains a permutation of order $p$.**

Suppose the proposition is true for groups whose order is the product of fewer than $n$ prime numbers, as we know it to be true by proposition (25) for groups of order $p$. We proceed to prove that it then is true also for groups whose order is the product of $n$ prime numbers.

[1] The Theorem of Cauchy was stated without proof by Galois.

Let $G$ be such a group. If it contains a subgroup whose index is prime to $p$, then the order of the subgroup is divisible by $p$ and the subgroup contains by assumption a permutation of order $p$.

Otherwise the index of every subgroup is divisible by $p$. By proposition (82) the number of permutations in any class of $G$ then is divisible by $p$, too, if it is not one. Also the sum of such numbers is divisible by $p$, because it is by proposition (81) equal to the order of $G$. Since the number for the class identity is one, we conclude that there must be more classes with just one permutation and altogether a multiple of $p$ such classes.

The permutations of these classes compose the central subgroup of $G$. This subgroup is Abelian by proposition (73) and, having an order equal to a multiple of $p$, contains by proposition (77) a permutation of order $p$.

Thus one subgroup of $G$ must contain a permutation of order $p$, which proves the theorem.

## §58. METACYCLIC GROUP

While we traced the composition-series of the symmetric group from that group down to identity and found that no symmetric group of degree more than four is soluble, we now reverse the procedure: tracing the composition-series from identity up to larger groups, we search for the largest soluble group within our reach.

But in doing so, we restrict our purpose in two ways:

(1) we confine our work to transitive groups

(2) we confine our work to groups of prime degree.

The soluble group next to identity then is the cyclic group

$$C = \{s\}$$

of order $p$ with

$$s = \begin{pmatrix} 1\,2 & . & . & . & p \\ 2\,3 & . & . & . & 1 \end{pmatrix} = (12 \; . \; . \; . \; p).$$

In an other notation, with the modulus $p$ understood, we write

$$s = \begin{pmatrix} z \\ z + 1 \end{pmatrix};$$

which means that every subscript

$$z = 1, 2, \; . \; . \; . \; , p$$

is replaced by the subscript $z + 1$ and $p$ by 1 since

$$p \equiv 0 (\text{mod } p).$$

In this notation we obviously have

$$s^2 = \binom{z}{z+2}, \quad s^3 = \binom{z}{z+3}, \quad \cdots;$$

so that all permutations in $C$ are of the form

$$s^\lambda = \binom{z}{z+\lambda}$$

with

$$\lambda = 1, 2, \quad . \; . \; . \; , p.$$

Any transitive group contains the cyclic as a subgroup, by the Theorem of Cauchy.[1] And the cyclic group immediately precedes identity in the composition-series of a transitive group, if the transitive group is soluble. For

(84) **a normal subgroup of a transitive group of prime degree is transitive, unless it is identity;**

otherwise a transitive group of prime degree $p$ would have to be imprimitive by proposition (65), but $p$ letters cannot divide into equal sets. It follows that every transitive group has a transitive composition-series, by which we mean a composition-series formed by transitive groups alone.

Hence we search for the largest group containing the cyclic as a normal subgroup. That group beyond the cyclic we call the **metacyclic group,**[2] using the term in its original and literal meaning, and we denote the group by $M$. The composition-series of the metacyclic group is

$$M \quad . \; . \; . \quad C \qquad 1.$$

Suppose that

$$t = \begin{pmatrix} 12 & . \; . \; . & p \\ ab & . \; . \; . & k \end{pmatrix}$$

is any permutation of $M$ which is not in $C$. To represent it in the other notation as

$$t = \binom{z}{\varphi(z)},$$

we need a function $\varphi(z)$ such that

$$\varphi(1) = a, \quad \varphi(2) = b, \quad . \; . \; . \; , \quad \varphi(p) = k.$$

[1] The order of a transitive group is a multiple of $p$, by proposition (60).

[2] The metacyclic group is denoted as **linear group** or **congruence group** when metacyclic is used in the sense of soluble, and these terms are applied to include the subgroups of the metacyclic group. This has to be remembered well if one is to understand the literature.

Such a function is given by the interpolation formula of Lagrange:[1]

$$\varphi(z) = \frac{af(z)}{(z-1)f'(1)} + \frac{bf(z)}{(z-2)f'(2)} + \cdots + \frac{kf(z)}{(z-p)f'(p)},$$

where

$$f(z) = (z-1)(z-2)\ldots(z-p)$$

and $f'(z)$ is the derivative of $f(z)$.

Since $C$ is normal in $M$, the transform of a permutation $s$ of $C$ by a permutation $t$ of $M$ is contained in $C$ and hence a power of $s$:

$$t^{-1}st = s^{\mu};$$

here

$$\mu \neq 1$$

because $s$ is not identity. But we have:

$$t^{-1} = \begin{pmatrix} \varphi(z) \\ z \end{pmatrix}$$

$$t^{-1}s = \begin{pmatrix} \varphi(z) \\ z+1 \end{pmatrix}$$

$$t^{-1}st = \begin{pmatrix} \varphi(z) \\ \varphi(z+1) \end{pmatrix},$$

while

$$s^{\mu} = \begin{pmatrix} z \\ z+\mu \end{pmatrix}$$

can be written in the form

$$s^{\mu} = \begin{pmatrix} \varphi(z) \\ \varphi(z)+\mu \end{pmatrix}$$

because $\varphi(z)$, just as $z$, stands for the subscripts

$$1, 2, \ldots, p.$$

It follows that

$$\varphi(z+1) \equiv \varphi(z) + \mu (\text{mod } p).$$

Putting now

$$\varphi(0) \equiv \varphi(p) \equiv \nu,$$

we have

$$\varphi(1) \equiv \nu + \mu$$
$$\varphi(2) \equiv \varphi(1) + \mu \equiv \nu + 2\mu$$
$$\cdots\cdots\cdots$$

and in general:

$$\varphi(z) \equiv \nu + z\mu (\text{mod } p).$$

[1] Given in §1.

Hence it appears that

$$t = \binom{z}{\mu z + \nu},$$

where

$$\mu = 1, 2, \ldots, p - 1$$
$$\nu = 1, 2, \ldots, p - 1, p.$$

Consequently there exist $p(p - 1)$ distinct permutations $t$ composing the group $M$, and we note:

(85) **The largest group of prime degree $p$ containing the cyclic group of degree and order $p$ as a normal subgroup is the metacyclic group of order $p(p - 1)$ with all permutations of the form**

$$\boxed{t = \binom{z}{\mu z + \nu}}.$$

The metacyclic group contains no circular permutations of order $p$ other than those in $C$, for any permutation

$$t = \binom{z}{\mu z + \nu}$$

with

$$\mu \neq 1$$

leaves unaltered that letter whose subscript is determined by the congruence[1]

$$\mu z + \nu \equiv z (\operatorname{mod} p).$$

It will be shown in §81 that there exists a number $g$ such that modulo $p$ its powers

$$g, g^2, \ldots, g^{p-1}$$

are the numbers

$$1, 2, \ldots, p - 1$$

in some order or other.

This makes it possible to represent the permutations of the metacyclic group in the form[2]

$$t = s^\lambda u^\mu,$$

[1] Cf. §80.

[2] $s^\lambda u^\mu = \binom{z}{z + \lambda}\binom{z}{g^\mu z} = \binom{z}{g^\mu(z + \lambda)}$ stands for the same permutations as $t = \binom{z}{\mu z + \nu}$.

where

$$s = \binom{z}{z+1}$$

$$u = \binom{z}{gz}.$$

A function $\psi$ belonging to $C$ is unaltered by $s^\lambda$ while $u$ converts

$$\psi \to \psi_u \to \psi_{u^2} \to \ . \ . \ . \to \psi_{u^{p-2}}.$$

All these functions are conjugate with $\psi$ in $M$ and belong to $C$ which is normal.

Hence the group $\Gamma$ of permutations between these functions, as caused by the permutations $t$ of $M$, is cyclic, and so is the factor-group $M/C$.[1] This group thus has a normal subgroup of any index dividing $p - 1$, by proposition (75), consequently $M$ has such a subgroup by proposition (68).

It follows that the metacyclic group is soluble; but we can prove even more:

(86) **The metacyclic group of prime degree is the largest transitive group of that degree which is soluble.**

For suppose that the metacyclic group $M$ or any of its transitive subgroups is contained in a larger group $G$ as a normal sub group. If $t$ is any permutation of $G$, it transforms the permuta tion $s$ of $M$ not into any permutation of $M$ but a power of $s$ because the transform is by proposition (30) a circular permuta tion of order $p$, and there are no such permutations outside $C$.

Therefore

$$t^{-1}st = s^\mu,$$

and we are back to the relation which gave us the metacyclic group.

The metacyclic group of degree three is

$$\binom{z}{z} \quad \binom{z}{z+1} \quad \binom{z}{z+2} \quad \binom{z}{2z} \quad \binom{z}{2z+1} \quad \binom{z}{2z+2},$$

or respectively

| | | | | | |
|---|---|---|---|---|---|
| 1 | (123) | (132) | (12) | (13) | (23), |

which is the symmetric group as we should expect it for the degrees three and four.

[1] Cf. proposition (67).

The metacyclic group of degree five is:

$$\binom{z}{z}\quad \binom{z}{z+1}\quad \binom{z}{z+2}\quad \binom{z}{z+3}\quad \binom{z}{z+4}$$
$$1,\quad (12345),\ (13524),\ (14253),\ (15432)$$

$$\binom{z}{2z}\quad \binom{z}{2z+1}\binom{z}{2z+2}\binom{z}{2z+3}\binom{z}{2z+4}$$
$$(1243),(1325),\quad (1452),\quad (1534),\quad (2354)$$

$$\binom{z}{3z}\quad \binom{z}{3z+1}\binom{z}{3z+2}\binom{z}{3z+3}\binom{z}{3z+4}$$
$$(1342),(1435),\quad (1523),\quad (2453),\quad (1254)$$

$$\binom{z}{4z}\quad \binom{z}{4z+1}\binom{z}{4z+2}\binom{z}{4z+3}\binom{z}{4z+4}$$
$$(14)(23),(15)(24),\ (25)(34),\ (12)(35),\ (13)(45).$$

The permutations

$$t = s^\lambda v^\xi,$$

where

$$s = \binom{z}{z+1}$$
$$v = \binom{z}{g^2 z}$$

and

$$\xi = 1, 2, \ldots, \frac{p-1}{2},$$

compose a group which is called **half-metacyclic.**

The permutation

$$s = (12 \ldots p)$$

is even, the permutation

$$u = (1g \ldots g^{p-2})$$

is odd because it leaves $x_p$ unaltered. The permutation $v$ is even: it is composed of two equal cycles, since p − 1 is divisible by 2 and $g^{p-1}$ equals the subscript we started the cycle with.[1] Hence we infer for a prime degree that

(87) **the metacyclic group is not contained in the alternating group, but the half-metacyclic group is.**

No odd power of $u$ can be an even permutation, because it has an odd number of odd cycles, $u$ itself being an odd permutation.

[1] Cf. Fermat's Theorem in §81.

But an even power of $u$ is a power of $v$; hence the half-metacyclic group contains all even permutations of the metacyclic group and

(88) **the half-metacyclic group is normal of index two in the metacyclic group**

by proposition (30).[1]

The half-metacyclic group of degree five is:

$$1,\ (12345),\ (13524),\ (14253),\ (15432),$$
$$(14)(23),\ (15)(24),\ (25)(34),\ (12)(35),\ (13)(45).$$

For this degree and

$$g = 2$$

we have

$$s = (12345),\ u = (1243),\ v = (14)(23).$$

A function unaltered by $s$ and $v$ is

$$\psi = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1.$$

It is converted by $u$ into

$$\psi' = x_1x_3 + x_3x_5 + x_5x_2 + x_2x_4 + x_4x_1,$$

also unaltered by $s$ and $v$. Both $\psi$ and $\psi'$ belong to the half-metacyclic group and are conjugate under the metacyclic group. The sum

$$\psi + \psi'$$

is symmetric, while the difference

$$\psi - \psi'$$

belongs to the half-metacyclic group and its square

$$\varphi = (\psi - \psi')^2$$

to the metacyclic group.

## NOTE ON ABSTRACT GROUP

The concept of a group originated with the theory of permutations, but reaching out as it grew it finally became disassociated from any concrete operations whatsoever while applying to all of them. In its abstract form it embodies a certain law of combination exhibited on abstract operations or **elements,** and around its abstract form clusters the modern theory of **finite groups.**

Elements of a set, to qualify for the theory of groups, must

(1) possess a law of combination

(2) satisfy the associative law

[1] Because it is identical with its transforms.

(3) contain the **identical element**[1]

(4) contain the **inverse** of every element.

A set of elements forms a **group** if

(1) the product of any two elements is an element of the set

(2) the set contains the inverse of every element.

The number of elements in a group is its **order.**

Groups are **commutative** or **non-commutative** depending on whether their elements are commutative or non-commutative.

Groups can be **isomorphic,** but simply isomorphic groups are identical representing the same law of combination.

A group is defined by a set of **generating elements** satisfying independent and consistent relations. For instance, the generating elements $a$, $b$, $c$ satisfying the relations

$$a^2 = 1,\ b^2 = 1,\ ab = ba = c$$

define a group of order four:

$$\Gamma = 1, a, b, c.$$

This we see from the multiplication table:

$$\begin{array}{c|ccc} 1 & a & b & c \\ \hline a & aa & ab & ac \\ b & ba & bb & bc \\ c & ca & cb & cc \end{array} \quad = \quad \begin{array}{c|ccc} 1 & a & b & c \\ \hline a & 1 & c & b \\ b & c & 1 & a \\ c & b & a & 1 \end{array},$$

for we obtain from the given independent relations the other ones:

$$\begin{aligned} ac &= a^2b = b,\ cb = ab^2 = a \\ bc &= b^2a = a,\ ca = ba^2 = b \\ cc &= abba = 1. \end{aligned}$$

The symmetry of the multiplication table around the main diagonal indicates that the group is commutative.

The theory of finite groups, developed from the theory of permutation-groups, remains essentially identical with it, for **Cayley's Theorem** states that

(89) **every group can be represented as permutation-group on its elements.**

Let the elements of a group $\Gamma$ be

$$a, b, c, \ldots$$

[1] Denoted by 1 if no confusion can arise; it is 0 in ordinary addition.

The combination of these elements with one of them, say $a$, obviously gives the same elements in the same or another order, so that

$$s_a = \begin{pmatrix} a & b & c & \ldots \\ aa & ba & ca & \ldots \end{pmatrix}$$

stands for a permutation. In the same sense we set

$$s_b = \begin{pmatrix} a & b & c & \ldots \\ ab & bb & cb & \ldots \end{pmatrix},$$

which can be written

$$s_b = \begin{pmatrix} aa & ba & ca & \ldots \\ aab & bab & cab & \ldots \end{pmatrix}$$

since the latter permutation, as the former, only replaces every element of $\Gamma$ by the same element multiplied with $b$.

Combining $s_a$ and $s_b$ we obtain

$$s_a s_b = \begin{pmatrix} a & b & c & \ldots \\ aa & ba & ca & \ldots \end{pmatrix}\begin{pmatrix} aa & ba & ca & \ldots \\ aab & bab & cab & \ldots \end{pmatrix} = \begin{pmatrix} a & b & c & \ldots \\ aab & bab & cab & \ldots \end{pmatrix} = s_{ab}$$

and see that the permutations $s_a$ and $s_b$ combine as the elements $a$ and $b$ do. Hence it follows that the groups

$$\begin{aligned} \Gamma &= a, \quad b, \quad c, \quad \ldots \\ G &= s_a, \quad s_b, \quad s_c, \quad \ldots \end{aligned}$$

are simply isomorphic and abstractly speaking identical, which proves the proposition.

For our example of an abstract group we find

$$\begin{aligned} \Gamma &= 1, \quad a, \quad b, \quad c \\ G &= 1, \ (1a)(bc), \ (1b)(ac), \ (1c)(ab) = V. \end{aligned}$$

# CHAPTER X

# DOMAIN

## §59. ALGEBRAIC DOMAIN

Numbers compose a **domain or field** if they combine into such numbers only as are among them when acted upon by the rational operations of arithmetic (except division by zero).

Acting upon numbers 1, the rational operations produce the domain formed by the rational numbers of arithmetic. It is commonly called the **domain of rational numbers** and denoted by (1).

Every domain[1] contains the number 1 as quotient of any number other than zero in the domain by itself, and contains therefore the domain (1) of rational numbers.

If we **adjoin** to the domain (1) a number $\alpha$ not contained in it, we obtain a larger domain denoted by $(\alpha)$; it includes (1) and $\alpha$ and everything that the rational operations can give acting upon $\alpha$ and the numbers of (1). If we adjoin to the domain $(\alpha)$ a number $\beta$ not contained in it, we obtain a still larger domain $(\alpha, \beta)$.

Whenever an implicit notation is expedient, we denote a domain by $\Omega$, a symbol which qualifies for this use by its very form, it would seem. A number in $\Omega$ we can denote by $\omega$. If we set

$$\Omega = (\alpha),$$

then

$$\Omega(\beta) = (\alpha, \beta).$$

Every number belongs to some domain. An integral function

$$f(x) = a_0x^n + a_1x^{n-1} + \ . \ . \ . \ + a_n$$

is considered as belonging to such a domain

$$(a_\nu)_1^n = (a_1, \ . \ . \ . \ , a_n)$$

as contains its coefficients and hence symmetric functions of its roots. Number and function are said to be **rational in a domain**

[1] Other than (0), which we do not regard as forming a domain properly speaking.

to which they belong, and a number also to be **rationally known** there.[1] It may be well to note that we mean by a domain to which a number or function belongs the smallest such domain, unless specified otherwise.

When we say in elementary algebra that a function $x^2 - 2$ is irreducible, we do not quite mean what we say: we only mean to say that it is irreducible in the domain (1) of rational numbers, for evidently the function

$$x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$$

is reducible in the domain $(\sqrt{2})$. And although the function

$$x^2 + 2 = (x + \sqrt{2}\,i)(x - \sqrt{2}\,i)$$

is irreducible in the domains (1) and $(\sqrt{2})$, it is reducible in the domain $(\sqrt{2}, i)$.

Thus, a function is reducible or irreducible only with reference to a domain. Any function becomes reducible if we adjoin a root of the function to its domain, and the linear function alone is not reducible any further.

We shall denote functions reducible in a domain preferably by large letters and functions irreducible so preferably by small letters.

An equation

$$F(x) = 0$$

is considered rational and reducible in the domain in which the function $F(x)$ is so.

If an equation

$$f(x) = a_0x^n + a_1x^{n-1} + \ . \ . \ . + a_n = 0$$

is rational but irreducible in the domain

$$\Omega = (a_i)_1^n,$$

it becomes reducible in the domain

$$\Omega(x_1) = (a_i, x_1)_1^n$$

which we obtain adjoining to $(a_i)_1^n$ the root $x_1$ of the equation; for there we can separate from $f(x)$ the factor $x - x_1$. Hence the equation can be solved in the domain

$$\Omega(x_i)_1^n = (a_i, x_i)_1^n,$$

where it is reducible to linear factors.

[1] In short, we shall say that a number or function is in $\Omega$, instead of rational in $\Omega$.

Domains that we obtain by adjoining to the domain $\Omega$ of an equation its algebraic roots[1]—which alone concern us here—are called **algebraic domains** on $\Omega$, and $\Omega$ is said to be a **subdomain** of those. Not every equation is reducible in an algebraic domain since not every equation has algebraic roots.[2]

For integral functions we note:

(90) **A function $f(x)$ which is rational but irreducible in the domain $\Omega$ has no common factor with another function $F(x)$ which is rational in $\Omega$ unless the entire function $f(x)$ is a factor of the function $F(x)$.**

This is obvious since the highest common factor of the functions $F(x)$ and $f(x)$ is computable from them by rational operations[3] and therefore belongs to their domain $\Omega$, while we assume that the function $f(x)$ has no factor rational in $\Omega$ other than itself.

It follows that the function $f(x)$ has none of its roots or all of them in common with the function $F(x)$, and we note in particular that it has all of them in common if it has one.

It also follows that an irreducible function never has a double root, since it cannot have a root in common with its derivative.

## §60. ALGEBRAIC DOMAIN, *Continued*

A root $x_1$ of an equation

$$F(x) = 0$$

rational[4] in the domain $\Omega$ is also a root of an irreducible equation

$$f(x) = 0$$

rational in $\Omega$. For $F(x)$, if not itself irreducible in $\Omega$, must have such a factor $f(x)$ containing the root $x_1$. If we set

$$f(x) = x^n + a_1x^{n-1} + \ . \ . \ . \ + a_n$$

with

$$a_0 = 1,$$

then $x_1$ is a root of a very definite equation rational and irreducible in $\Omega$.

It is the degree of this equation which determines the **degree** of the algebraic domain $\Omega\ (x_1)$.

[1] We mean algebraically computable roots. Cf. §1.

[2] Cf. §82, end.

[3] By proposition (2).

[4] This refers to the equation.

We now prove for any root $x_1$ that

(91) **the algebraic domain $\Omega(x_1)$ consists of all integral functions of $x_1$ which are rational in $\Omega$.**

Since any number $\omega_1$ in $\Omega(x_1)$ can be obtained from $x_1$ and the numbers in $\Omega$ by rational operations, it is expressible as function of $x_1$ rational in $\Omega$:

$$\omega_1 = \frac{\varphi(x_1)}{\psi(x_1)},$$

where $\varphi$ and $\psi$ are integral functions. Since $\psi(x_1)$ cannot be zero, $\psi(x)$ has no factor in common with $f(x)$ which is irreducible,[1] and by proposition (4) we may set

$$Rf + r\psi = \varphi,$$

where $R$ and $r$ are integral functions rational in $\Omega$ and $r$ is of degree less than $n$.

If now

$$x = x_1,$$

then

$$f = 0$$

and

$$\frac{\varphi(x_1)}{\psi(x_1)} = r(x_1).$$

Hence any number in $\Omega(x_1)$ is expressible as integral rational function of $x_1$:

$$\omega_1 = r(x_1).$$

This function is unique as from another relation

$$\omega_1 = \rho(x_1)$$

would follow that the equation

$$r(x) - \rho(x) = 0$$

of degree less than $n$ and rational in $\Omega$ is satisfied by

$$x = x_1.$$

But then all other roots of $f(x)$ would have to satisfy this equation by proposition (90), for $f(x)$ is rational and irreducible in $\Omega$. This is impossible since $f(x)$ is of degree $n$, which completes our proof.

[1] By proposition (90).

## §61. CONJUGATE DOMAINS

No root $x_1$ of the irreducible function $f(x)$ of degree $n$ in the domain $\Omega$ is rational in $\Omega$, since the function $f(x)$ has no factor $x - x_1$ rational in $\Omega$. Adjoining the different roots $x_i$ of $f(x)$ to $\Omega$, we obtain the **conjugate domains** on $\Omega$:

$$\Omega(x_1),\ \Omega(x_2),\ .\ .\ .\ ,\ \Omega(x_n),$$

and we note that they all are of degree $n$. The number of conjugate domains thus equals the degree of any one.

Such numbers in the conjugate domains as we obtain by giving to the variable $x$ in the function $r(x)$ the values $x_i$:

$$\omega_1 = r(x_1),\ \omega_2 = r(x_2),\ .\ .\ .\ ,\ \omega_n = r(x_n),$$

are called **conjugate numbers;** there is one in each conjugate domain.

It is clear that

(92) **every number $\omega_1$ in an algebraic domain of degree $n$ on $\Omega$ is a root of an equation**

$$\boxed{\Phi(\omega) = (\omega - \omega_1)(\omega - \omega_2)\ .\ .\ .\ (\omega - \omega_n) = 0}$$

**of degree $n$ whose other roots are the numbers conjugate with $\omega_1$ and which is rational in $\Omega$.**

For the coefficients of $\Phi(\omega)$ are symmetric in the $x_i$ since any permutation on the $x_i$ only interchanges the $\omega_i$.

If $\Phi(\omega)$ is reducible in $\Omega$, as it may be, it must have there irreducible factors since the $\omega_i$ are not in $\Omega$. If $\varphi(\omega)$ is one such factor, the equation

$$\varphi(\omega) = 0$$

in $\Omega$ must be satisfied by some number $\omega_i$, say $\omega_1$. From

$$\varphi(\omega_1) = 0,$$

which is the same as

$$\varphi[r(x_1)] = 0,$$

it then appears that $x_1$ is a root of the equation

$$\varphi[r(x)] = 0$$

in $\Omega$. Since a root of the irreducible equation

$$f(x) = 0$$

in $\Omega$ cannot satisfy another equation of the same domain unless all its roots do so,[1] we then have

$$\varphi[r(x_i)] = 0,$$

which is the same as $[i = 1, \ldots, n$

$$\varphi(\omega_i) = 0.$$

Hence $\varphi(\omega)$ has all the roots of $\Phi(\omega)$ and consequently

$$\Phi(\omega) = \varphi(\omega).$$

But this is completely true only if all numbers $\omega_i$ are distinct. If among them the numbers

$$\omega_1, \ldots, \omega_m$$

alone are distinct while any other number $\omega_i$ equals one of these, we still assume that the function

$$\varphi(\omega) = (\omega - \omega_1) \ldots (\omega - \omega_m)$$

is rational and irreducible in $\Omega$; but the function $\Phi(\omega)$ then is reducible in $\Omega$ since it has the factor $\varphi(\omega)$. Any other irreducible factor of $\Phi(\omega)$ must be equal to $\varphi(\omega)$,[1] having some root $\omega_i$ in common with $\varphi(\omega)$, whence

$$\Phi(\omega) = \varphi^k(\omega),$$

where

$$km = n.$$

Thus,

(93) **the function $\Phi(\omega)$ is either irreducible in $\Omega$ or else some power of an irreducible function, while conjugate numbers $\omega_i$ are either all unlike or else divide into equal sets of unlike numbers.**

A number of an algebraic domain $\Omega(x_1)$ different from its conjugates is called a **primitive number** of that domain and may be denoted by $\theta_1$. We observe that it is a root of an irreducible equation in $\Omega$ which has the same degree as the domain $\Omega(x_1)$ has on $\Omega$.

If the numbers $\omega_i$ are **imprimitive,** dividing into equal sets of $m$ unlike numbers, a number $\omega_1$ evidently is primitive in an algebraic domain of degree $m$ on $\Omega$; whence it is rational in $\Omega$ if the $\omega_i$ are all alike.

[1] By proposition (90).

§62. **CONJUGATE DOMAINS,** *Continued*

Let $\omega_1$ be any number in the algebraic domain $\Omega(x_1)$ of which $\theta_1$ is a primitive number, and let their conjugate values be

$$\omega_1 = r(x_1),\ \omega_2 = r(x_2),\ \ldots,\ \omega_n = r(x_n)$$
$$\theta_1 = \rho(x_1),\ \theta_2 = \rho(x_2),\ \ldots,\ \theta_n = \rho(x_n).$$

The function

$$\frac{\omega_1}{\theta - \theta_1} + \frac{\omega_2}{\theta - \theta_2} + \ldots + \frac{\omega_n}{\theta - \theta_n}$$

of $\theta$ is rational in $\Omega$ since it is symmetric in the $x_i$, any permutation on the $x_i$ only interchanging its terms. Multiplying it by the function

$$\varphi(\theta) = (\theta - \theta_1)(\theta - \theta_2)\ \ldots\ (\theta - \theta_n)$$

which also is rational in $\Omega$, we obtain

$$\varphi(\theta)\left(\frac{\omega_1}{\theta - \theta_1} + \frac{\omega_2}{\theta - \theta_2} + \ldots + \frac{\omega_n}{\theta - \theta_n}\right) = \chi(\theta)$$

which is an integral function rational in $\Omega$ and of degree $n - 1$ in $\theta$ if we expand it.

Setting in this function

$$\theta = \theta_1,$$

we find that

$$\omega_1 = \frac{\chi(\theta_1)}{(\theta_1 - \theta_2)\ \ldots\ (\theta_1 - \theta_n)} = \frac{\chi(\theta_1)}{\varphi'(\theta_1)},$$

where $\varphi'(\theta)$ is the derivative of $\varphi(\theta)$ and as such an integral function rational in $\Omega$; $\varphi'(\theta_1)$ cannot be zero since the $\theta_i$ are all unlike.

Thus we proved that

(94) **every number $\omega_1$ of an algebraic domain $\Omega(x_1)$ is expressible as a function rational in $\Omega$ of any primitive number $\theta_1$ in $\Omega(x_1)$:**

$$\boxed{\omega_1 = R(\theta_1)}.$$

This obviously means that the domains $\Omega(\theta_1)$ and $\Omega(x_1)$ are identical:

$$\Omega(\theta_1) = \Omega(x_1).$$

The primitive number $\theta_1$ is a root of an irreducible equation in $\Omega$ of degree $n$ while the imprimitive number $\omega_1$ is a root of an irreducible equation in $\Omega$ of degree less than $n$.[1] Hence $\theta_1$

[1] By proposition (93); cf. §61, end.

cannot be rational in $\Omega(\omega_1)$ and the latter is a subdomain of $\Omega(\theta_1)$ or $\Omega(x_1)$:

$$\Omega(\omega_1) < \Omega(x_1).$$

An algebraic domain on $\Omega$ containing no imprimitive numbers[1] is called a **primitive domain**; it certainly is primitive if its degree is prime, since its numbers then cannot divide into sets. When not primitive, an algebraic domain is called **imprimitive.** A primitive domain on $\Omega$ contains no other subdomain than $\Omega$.

Proposition (94) seems to recall the Theorem of Lagrange: there, functions of one group rationally expressible in terms of each other; here, functions of one domain expressible so. We feel that the circle of our investigation is going to close.[2]

## §63. NORMAL DOMAIN

If an algebraic domain of an irreducible equation is identical with its conjugate domains, it is called a **normal or invariant domain,** and the equation is called a **normal or invariant equation.** It appears that a normal equation is irreducible and has roots which are rationally expressible in terms of each other.

Suppose that the equation

$$f(x) = 0$$

of degree $n$, rational and irreducible in $\Omega$, is normal; then

$$\Omega(x_1) = \Omega(x_2) = \ . \ . \ . \ = \Omega(x_n).$$

Let $\theta_1$ be a primitive number in $\Omega(x_1)$ and have the conjugate values

$$\theta_1, \theta_2, \ . \ . \ . \ , \theta_n.$$

Since

$$\Omega(\theta_i) = \Omega(x_i)$$

by proposition (94), we have

$$\Omega(\theta_1) = \Omega(\theta_2) = \ . \ . \ . \ = \Omega(\theta_n),$$

and consequently the $\theta_i$ are roots of the equation

$$g(\theta) = 0$$

also normal of degree $n$ in $\Omega$.

[1] Outside those in $\Omega$, of course.

[2] The proof of proposition (94) can be applied to Lagrange's Theorem, and the proof given there can be applied here; but the present proof adds an elegant tool to the mathematical outfit of the student. Compare §69.

Any one root of a normal equation in $\Omega$, and of such an equation alone, is expressible as a function rational in $\Omega$ of any other root and we may set

$$\theta_1 = R_1(\theta_1),\ \theta_2 = R_2(\theta_1),\ \ldots,\ \theta_n = R_n(\theta_1).$$

If on these functions we perform a **substitution**[1] replacing $\theta_1$ by $\theta_k$, we obtain as result the same functions in another order because from

$$g[R_i(\theta_1)] = 0$$

follows[2] that every root of the irreducible equation

$$g(\theta) = 0$$

in $\Omega$ satisfies the equation

$$g[R_i(\theta)] = 0$$

in $\Omega$, whence with

$$R_i(\theta_1)$$

also

$$R_i(\theta_k)$$

is a root of

$$g(\theta) = 0.$$

And no two roots $R_i(\theta_k)$ and $R_j(\theta_k)$ are alike since from

$$R_i(\theta_k) - R_j(\theta_k) = 0$$

would follow as before that also

$$R_i(\theta_1) - R_j(\theta_1) = 0,$$

which is untrue.

The substitution that we performed on the $\theta_i$ we denote by

$$\sigma_k = (\theta_1\theta_k).$$

It converts any root

$$\theta_i = R_i(\theta_1)$$

into a root $\theta_j$ defined by the relation

$$\theta_j = R_i(\theta_k) = R_i[R_k(\theta_1)] = R_j(\theta_1),$$

so that of two roots like $\theta_i$ and $\theta_j$ we may choose one at pleasure while the other is fixed by the substitution.

Any number

$$\omega_1 = \varphi(\theta_1)$$

rational in $\Omega(\theta_1)$ is converted by the substitution $\sigma_k$ into

$$\omega_k = \varphi(\theta_k).$$

[1] Compare §9.

[2] By proposition (90).

As $\omega_1$ is rational also in $\Omega(\theta_i)$, we may set

$$\omega_1 = \psi(\theta_i),$$

and this is converted by $\sigma_k$ into

$$\omega_k = \psi(\theta_j),$$

whence

$$\sigma_k = (\theta_1\theta_k) = (\theta_i\theta_j).$$

The substitutions $(\theta_i\theta_j)$ are called **substitutions of the normal domain** $\Omega(\theta_1)$, and since they can be expressed in the form $(\theta_1\theta_k)$, the normal domain has as many different substitutions as its degree indicates, which is $n$.

Every one of these substitutions converts a number $\omega_1$ of the normal domain $\Omega(\theta_1)$ into a number of the same domain, and no two numbers into the same number. If a number $\omega_1$ remains unaltered by every substitution of the normal domain, it is identical with its conjugates and therefore rational in $\Omega$.

The substitutions of the normal domain $\Omega(\theta_1)$ are qualified elements composing a group in the sense of the note to chapter nine. For

$$\sigma_i\sigma_k = \sigma_j$$

since

$$(\theta_1\theta_i)(\theta_i\theta_j) = (\theta_1\theta_j),$$

if we arrange it so that the first letter in the second substitution is identical with the second letter in the first substitution, and the other qualifications are readily verified.

It appears that

(95) **the substitutions of a normal domain compose a group**

$$\boxed{\langle\Gamma\rangle = (\theta_1\theta_i)_1^n},$$

and this leads us back to the theory of equations.

# CHAPTER XI

# THEORY OF GALOIS

## §64. SPECIAL EQUATION

Disappointed in our effort to solve the general equation, we now turn to special equations, that is to say equations with numerical coefficients. Therefore it has to be kept in mind firmly that in all what follows we are dealing with numerical values, unless specified otherwise.

Lagrange's plan of solving the general equation falls short if applied to numerical equations whenever we strike conjugate functions that are equal. Thus, for the equation

$$x^3 - 2 = 0$$

with the roots

$$x_1 = 2^{1/3},\ x_2 = \omega 2^{1/3},\ x_3 = \omega^2 2^{1/3}$$

the conjugate functions

$$\begin{aligned} \psi_1 &= x_2 x_3 - x_1^2 = 0 \\ \psi_2 &= x_1 x_3 - x_2^2 = 0 \\ \psi_3 &= x_1 x_2 - x_3^2 = 0 \end{aligned}$$

would not permit a computation by the formula

$$\varphi_1 = \frac{R(\psi_1)}{\Delta_\psi}$$

of Lagrange's Theorem, since the denominator becomes zero. Likewise, for the equation

$$x^4 + 1 = 0$$

with the roots

$$x_1 = i^{1/2},\ x_2 = i^{3/2},\ x_3 = i^{5/2}\ x_4 = i^{7/2}$$

the conjugate functions

$$\begin{array}{llll} \chi_1 = x_1^2 & \chi_2 = x_2^2 & \chi_3 = x^{\ 2} & \chi_4 = \chi_4^2 \\ \quad = i, & \quad = -i, & \quad = i, & \quad = -i. \end{array}$$

would not permit such a computation.

Furthermore, the permutations that leave $\psi_1$ unaltered in value, either because they leave it unaltered in formal value:

$$1, (23),$$

or because they convert it into $\psi_2$ and $\psi_3$:

$$(12), (13), (123), (132),$$

they do form a group. But those that leave $\chi_1$ unaltered in value, either because they leave it unaltered in formal value:

$$1, (23), (24), (34), (234), (243),$$

or because they convert it into $\chi_3$:

$$(13), (13)(24), (132), (134), (1324), (1342),$$

they do not form a group. Hence, passing to numerical values, we cannot say any longer that permutations which leave the value of a function unaltered compose a group.

To become applicable to special equations, Lagrange's plan had to be modified, and this work was done by Galois.[1] Lagrange's special plan of solving general equations young Galois replaced by his general plan of solving special equations, and a touch of supreme genius marks his achievement: it is an epic of human glory

"not ill-befitting men that strove with gods."

### §65. GALOISIAN FUNCTION

Permutations that leave a function numerically unaltered do not in general form a group; but when they concur with those that leave the function also formally unaltered, they evidently do so. Discarding functions that remain numerically unaltered by permutations which do not leave them formally unaltered, Galois taught us to construct functions that belong to a group both formally and numerically; and such functions we call **Galoisian functions.**

Let

$$F(x) = a_0x^n + a_1x^{n-1} + \ . \ . \ . + a_n = 0$$

be a special equation rational in the domain

$$\Omega = (a_\iota)_1^n$$

[1] Galois lived 1811–1832. He was seventeen when he obtained his first beautiful results in mathematics, at nineteen he had divined the nature of equations, and he was twenty when a duel ended his stormy life.

of its coefficients, or any wider domain $\Omega$, and let us impose on it the condition that it has no double roots. Evidently this condition does not limit the range of the theory which is to be built up, for double roots of a function are intimated by its vanishing discriminant and readily found and eliminated as roots of the common factor which the function has with its derivative.

Then,

(96) **it is always possible to construct in the domain $\Omega$ of an equation a Galoisian function of its roots $x_i$ which belongs to the group identity on the $x_i$.**

Altered by every permutation on the $x_i$ both formally and numerically, such a function $v_1$ takes under the symmetric group

$$S = 1, s_2, \ldots, s_{n!}$$

on the $x_i$ the conjugate values

$$v_1, v_2, \ldots, v_{n!}$$

and may be called an **elementary Galoisian function** for the given equation.

Galois puts

$$v_1 = u_1x_1 + u_2x_2 + \ldots + u_nx_n$$

with indeterminate $u_i$; for the integral rational function[1]

$$\prod_{j \lessgtr k}(v_j - v_k) = J(a_i, u_i)_1^n \qquad [j, k = 1, \ldots, n!$$

does not vanish identically,[2] so that we may assign to the $u_i$ integral numerical values that do not make $\prod$ vanish, that is to say do not make two values $v_j$ equal.

It follows that

(97) **it is always possible to construct in the domain $\Omega$ of an equation a Galoisian function of its roots $x_i$ which belongs to any group on the $x_i$,**

both formally and numerically.

Suppose such a group is

$$G = 1, s_2, \ldots, s_r$$

converting the elementary Galoisian function $v_1$ into

$$v_1, v_2, \ldots, v_r,$$

[1] Integral and rational in both the $a_i$ and the $u_i$, for symmetric in the $x_i$.

[2] Because the $v_j$ are distinct by §16, the $u_i$ being indeterminate quantities.

so that like subscripts identify the conjugate values into which the permutations of $G$ convert $v_1$. Galois obtains a function which satisfies the proposition by putting[1]

$$g(v) = (v - v_1)(v - v_2) \ . \ . \ . \ (v - v_r),$$

where $v$ is a rational number properly to be chosen but for the time indeterminate.

For a permutation $s_k$ of $G$ only interchanges the $v_j$, since applied to $v_j$ it gives the same result as $s_j s_k$ applied to $v_1$ and hence a conjugate value which is in the set as $s_j s_k$ is in $G$. While a permutation of $G$ thus leaves $g(v)$ unaltered, a permutation not in $G$ converts the $v_j$ of the set into such as are outside and alters $g(v)$.

It will be observed that functions as Galois taught us to construct are integral functions.

Applying Lagrange's Theorem to Galoisian functions, we evidently need not fear trouble:

(98) **If a rational function $\varphi$ in the roots $x_i$ of a special equation remains formally unaltered by all those permutations on the $x_i$ that leave a Galoisian function $g$ of the $x_i$ unaltered, then the function $\varphi$ is rationally expressible in terms of the function $g$;**

here as rational is considered the domain $\Omega$ of the equation.

For the proposition is true by virtue of Lagrange's Theorem if we take the $x_i$ to be indeterminate quantities:

$$\varphi = \frac{R(g)}{\Delta_g}.$$

On replacing the indeterminate $x_i$ by numerical values we meet no difficulty, since the discriminant in the denominator cannot be zero.

In particular, any function $\varphi$ is rationally expressible in terms of the elementary Galoisian function $v_1$:

$$\varphi(x_i) = R(v_1).$$

As this relation is an identity in the $x_i$, both members remain identical when we operate a permutation $s$ on the $x_i$. This gives

$$\varphi_s(x_i) = R(v_s),$$

[1] To express that $g(v)$ is a function also of the $x_i$ we denote it by $g(v|x_i)$.

the subscript $s$ identifying the permutation that converts $\varphi_1$ into $\varphi_s$ and $v_1$ into $v_s$. Hence we note:

(99) **If $\varphi(x_i)$ is a rational function of $v_1$, then $\varphi_s(x_i)$ is the same function of $v_s$, the $v_i$ being elementary Galoisian functions in the $x_i$ and $s$ a permutation on the $x_i$.**

The proposition holds true for the roots $x_i$ themselves:

$$x_1 = R_1(v_1),\ x_2 = R_2(v_1),\ \ldots,\ x_n = R_n(v_1),$$

or

$$x_1 = R_1(v_1),\ x_2 = R_1(v_2),\ \ldots,\ x_n = R_1(v_n)$$

when for instance

$$s_2 = (12),\ \ldots,\ s_n = (1n).$$

This recalls to our mind that a special equation is solved when we know a Galoisian function $v_1$ of its roots belonging to identity.

## §66. GALOISIAN RESOLVENT

The function

$$G(v) = (v - v_1)(v - v_2)\ \ldots\ (v - v_{n!})$$

which belongs to the symmetric group on the $x_i$ we call a **complete Galoisian function** for the special equation

$$F(x) = 0.$$

It may happen that such a complete Galoisian function $G(v)$ is reducible in the domain $\Omega$ of the equation—which evidently implies that there exist functions[1] of the $x_i$ not symmetric and yet rational in $\Omega$. If $G(v)$ is reducible, it must have an irreducible factor $g(v)$ since we assume that the equation cannot be solved in $\Omega$; for when it can, there is no problem left to be considered.

The function $g(v)$, still rational in $\Omega$ but not further reducible there, must vanish for at least one value $v_i$, say $v_1$. If it has more roots, let them be

$$v_1,\ v_s,\ \ldots,\ v_t,$$

subscripts again identifying the permutations on the $x_i$ which applied to $v_1$ give its conjugate values. The function

$$g(v) = (v - v_1)(v - v_s)\ \ldots\ (v - v_t)$$

is called a **primary Galoisian function** and the equation

$$g(v) = 0$$

a **Galoisian resolvent** for the given equation.

[1] But they can exist even though $G(v)$ be irreducible, as will appear in §70.

If the complete Galoisian function $G(v)$ is irreducible in $\Omega$, it is itself a primary Galoisian function and

$$G(v) = 0$$

a Galoisian resolvent. Furthermore it is to be noted that also an equation

$$g(\psi) = 0$$

often is called a Galoisian resolvent, when $\psi$ is any function numerically belonging to identity, not necessarily a Galoisian function belonging to identity both formally and numerically.

For a normal equation we may set

$$v_i = x_i,$$

because a normal equation is irreducible and its roots $x_i$ belong to identity being functions rational in $\Omega$ of each other;[1] identity is the only permutation which the groups of the $x_i$ can have in common.

Again, since all roots $v_i$ of a Galoisian resolvent belong to identity, they are expressible by proposition (98) as functions rational in $\Omega$ of $v_1$. This permits to conclude that

(100) **every Galoisian resolvent is a normal equation, and every normal equation is its own Galoisian resolvent;**

further that

(101) **the domain $\Omega(v_1)$ is a normal domain if $v_1$ is an elementary Galoisian function for a special equation in $\Omega$.**

Also the roots $x_i$ are expressible as functions rational in $\Omega$ of $v_1$, while we have

$$v_1 = u_1x_1 + u_2x_2 + \ldots + u_nx_n.$$

Hence it appears that

$$\Omega(v_1) = \Omega(x_i)_1^n,$$

which is to say that

(102) **the adjunction of an elementary Galoisian function for a special equation is equivalent to the adjunction of all its roots.**

Thus Galois has taught us to construct a normal equation for any given equation and for its domain an algebraic domain which is normal.

[1] Being rational functions of each other, they must belong to the same group, by proposition (98) which presents a necessary and sufficient condition; compare §36.

## §67. GALOISIAN GROUP

The permutations on the $x_i$ converting a root $v_1$ of

$$G(v) = (v - v_1)(v - v_2) \ . \ . \ . \ (v - v_{n!}) = 0$$

into all its roots evidently compose a group. Galois discovered this always to be true for his resolvent:

(103) **The permutations on the $x_i$ converting a root $v_1$ of the Galoisian resolvent**

$$\boxed{g(v) = (v - v_1)(v - v_s) \ . \ . \ . \ (v - v_t) = 0}$$

**into all its roots compose a group.**

We call it the **Galoisian group** or simply **group of the equation**

$$F(x) = 0$$

in $\Omega$ and denote it by $G$ or $\langle G \rangle$.

Consider the root $v_s$ of the Galoisian resolvent. It is expressible as function rational in $\Omega$ of the root $v_1$:

$$v_s = R_s(v_1),$$

so that

$$g(v_s) = g[R_s(v_1)] = 0.$$

But also

$$g(v_1) = 0,$$

whence $v_1$ is a root of two equations in $\Omega$:

$$g(v) = 0$$

and

$$g[R_s(v)] = 0.$$

Since the first equation is irreducible in $\Omega$, all its roots must satisfy the second equation as one does.[1] In particular, the root $v_t$ must do so; this gives

$$g[R_s(v_t)] = 0,$$

whence $R_s(v_t)$ is a root of

$$g(v) = 0.$$

To find the significance of this root, we apply to the identity

$$v_s = R_s(v_1)$$

in the $x_i$ the permutation $t$ and obtain[2]

$$v_{st} = R_s(v_t).$$

[1] By proposition (90).

[2] By proposition (99).

Hence it appears that $v_{st}$ is among the roots of the Galoisian resolvent and therefore $st$ among the permutations converting $v_1$ into those roots.

This proves that the permutations on the $x_i$ converting $v_1$ into

$$v_1, \ v_s, \ . \ . \ . \ , v_t$$

compose a group, the Galoisian group

$$G = 1, \ s, \ . \ . \ . \ , t$$

of the given equation. Its order is equal to the degree of the Galoisian resolvent for that equation.

It is clear that the primary Galoisian function $g(v)$ belongs to the group $G$ of the equation; for the permutations of $G$ do nothing more than interchange its roots, and they are the only permutations which do so.

Since any two roots $v_i$ of the Galoisian resolvent are interchanged by just one permutation of the Galoisian group[1] and just one substitution of the normal domain, we obtain a beautiful result:

(104) **The Galoisian group of a special equation and the substitution group of its normal domain are simply isomorphic and abstractly identical:**

$$\boxed{\langle G \rangle = \langle \Gamma \rangle}.$$

This links up the theory of groups with the theory of domains.

## §68. PROPERTIES OF GALOISIAN GROUP

The Galoisian group of the special equation

$$F(x) = 0$$

rational in the domain of its coefficients or any wider domain has two fundamental properties which we formulate referring rationality to the domain $\Omega$ of the equation.

(105) **Property (1): Every rational function of the $x_i$ which has a rational value remains numerically unaltered under the Galoisian group.**

For let such a function be

$$\varphi_1(x_i) = \omega,$$

[1] Only the permutation $s$ converts $v_1$ into $v_s$, for instance.

where $\omega$ is a number rational in $\Omega$. Expressing the function rationally in terms of

$$v_1 = u_1x_1 + u_2x_2 + \ldots + u_nx_n,$$

which we can do by proposition (98), we set

$$\varphi_1 = R(v_1).$$

Now we apply to this identity in the $x_i$ the permutations of the Galoisian group and obtain[1]

$$\varphi_s = R(v_s)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\varphi_t = R(v_t).$$

Since the rational equation

$$R(v) = \omega$$

is satisfied by the root $v_1$ of the rational and irreducible resolvent

$$g(v) = 0,$$

it is satisfied by every root of it. Therefore we have

$$R(v_1) = R(v_s) = \ldots = R(v_t) = \omega$$

and consequently

$$\varphi_1 = \varphi_s = \ldots = \varphi_t = \omega,$$

which proves the proposition.

If a permutation $s'$ leaves every rational function of the $x_i$ which has a rational value numerically unaltered, then with

$$g(v_1) = 0$$

which we know to be true also

$$g(v_{s'}) = 0.$$

Hence $v_{s'}$ is a root of the Galoisian resolvent and consequently $s'$ a permutation of the Galoisian group. This means that

(106) **the Galoisian group is the largest group under which every rational function of the $x_i$ which has a rational value remains numerically unaltered.**

Since a rational equation

$$\varphi(x_i) = \omega$$

can be written in the form

$$\chi(x_i) = \varphi(x_i) - \omega = 0,$$

[1] By proposition (99).

so that the function $\chi(x_i)$ has a value rational in any domain, it follows from property (1) that

(107) **every rational equation in the $x_i$ remains true when operated on by the permutations of the Galoisian group;**

or permits those permutations, as we say.

The other property is

(108) **Property (2): Every rational function of the $x_i$ which remains numerically unaltered under the Galoisian group has a rational value.**

Assuming that

$$\varphi_1 = \varphi_s = \ . \ . \ . \ = \varphi_t,$$

we can set

$$\varphi_1 = \frac{1}{r}[R(v_1) + R(v_s) + \ . \ . \ . \ + R(v_t)],$$

if $r$ denotes the order of the Galoisian group. Since the coefficients of the Galoisian resolvent have rational values and every symmetric function of its roots is rationally expressible in terms of its coefficients, the symmetric function

$$R(v_1) + R(v_s) + \ . \ . \ . \ + R(v_t)$$

of its roots has a rational value, which proves the proposition.

If some group

$$G' = 1, s', \ . \ . \ . \ , t'$$

possesses property (2), the coefficients of the function

$$g'(v) = (v - v_1)(v - v_{s'}) \ . \ . \ . \ (v - v_{t'})$$

have rational values by assumption, for they remain numerically unaltered under $G'$. But being rational and having a root $v_1$ in common with the rational and irreducible function $g(v)$, the function $g'(v)$ must contain all its roots and consequently $G'$ all the permutations of the Galoisian group. This means that

(109) **the Galoisian group is the smallest group such that every rational function of the $x_i$ which remains under it numerically unaltered has a rational value.**

Since the Galoisian group is the largest group possessing property (1) and the smallest group possessing property (2), it follows that

(110) **the Galoisian group of an equation is uniquely defined by its fundamental properties.**

Now it is clear that the group of an equation does not depend on our choice of $v_1$ among the roots of the complete Galoisian function: any irreducible factor of

$$G(v) = g(v) \cdot g'(v) \dots$$

may serve as primary Galoisian function and must yield the same group. The group of the equation does not even depend on the construction of $v_1$ which is possible in any number of ways.

## §69. PLAN OF GALOIS

The coefficients of a Galoisian resolvent for a special equation are rationally known while any one root solves the equation. Therefore the plan for solving a special equation depends on the composition-series not of the symmetric but of the Galoisian group for the equation, as the coefficients of the resolvent belong to that group and every root belongs to identity.

The Galoisian group takes in the theory of special equations the same place which the symmetric group has in the theory of general equations, and the permutations of the symmetric group that are outside the Galoisian group are completely ignored in the theory of special equations.

In this theory we can make use of any rational functions remaining numerically unaltered by the permutations of a group, they do not have to be Galoisian; but these establish the possibility of constructing such functions and link up the plan of Galois with the plan of Lagrange.

Let $G$ be the Galoisian group of the special equation

$$F(x) = 0$$

rational in $\Omega$. In the theory of special equations a function of the roots $x_i$ is said to belong to a subgroup $H$ of $G$ when it remains numerically unaltered by all those, and only those, permutations of $G$ which are in $H$; so that also in the theory of special equations

(111) **to every group on the $x_i$ belongs a rational function of the $x_i$.**

Conversely, the permutations that leave a rational function unaltered compose a group, so that also in the theory of special equations

(112) **every rational function of the $x_i$ belongs to a group on the $x_i$.**

To prove this, let $s$ and $s'$ be two permutations of $H$ leaving a rational function $\psi_1$ of the $x_i$ unaltered, whence

$$\psi_1 = \psi_s = \psi_{s'}.$$

Then with

$$\psi_{s'} = \psi_1$$

also

$$\psi_{s's} = \psi_s,$$

by proposition (107). But from

$$\psi_{s's} = \psi_s = \psi_1$$

follows that the permutation $s's$ leaves the function $\psi_1$ unaltered; consequently it is contained in the same group $H$ with the permutations $s$ and $s'$.

Also in the theory of special equations the function $\psi_1$ takes under the partitions

$$H, Ht, \ldots$$

of $G$ the conjugate values

$$\psi_1, \psi_t, \ldots$$

For with

$$\psi_s = \psi_1$$

also

$$\psi_{st} = \psi_t,$$

by proposition (107). Hence all propositions in the theory of general equations which follow from this are readily verified for special equations. In particular:

(113) **If a group has a subgroup of index $j$, a rational function belonging to the subgroup takes $j$ conjugate values under the group which belong to conjugate subgroups.**

Also in the theory of special equations

(114) **the conjugate values of a rational function under a group to whose subgroup the function belongs are roots of a resolvent equation whose degree equals the index of the subgroup in the group and whose coefficients belong to the group.**

If the resolvent is

$$r(\psi) = (\psi - \psi_1) \ldots (\psi - \psi_t) = 0,$$

a permutation $t'$ of the group only interchanges its roots since any permutation $tt'$ is contained in a partition of the group.

Conjugate values $\psi_i$ of a function are roots of a resolvent equation in Lagrange's sense of the term, an ordinary resolvent so to say, well to be distinguished from a Galoisian resolvent.

Any rational function of the $x_i$ which belongs to the group $G$ is a number in $\Omega$ by property (2) of $G$. It follows for the domain $\Omega$ that

(115) **the conjugate values which a rational function takes under the group of a special equation are roots of a rational and irreducible resolvent.**

Were the resolvent reducible, the Galoisian group would not be the smallest group such that rational functions of the $x_i$ remaining under it numerically unaltered have a rational value.

The proposition is true for general equations if we replace the Galoisian group by the symmetric group. That the resolvent then is irreducible will be proved in §70.

Lagrange's Theorem is in the theory of special equations replaced by the **Theorem of Lagrange-Galois,**[1] which we formulate referring rationality to the domain $\Omega$ of the equation:

(116) **If a rational function $\varphi_1$ in the roots $x_i$ of a special equation remains numerically unaltered by all those permutations of the Galoisian group for the equation which leave another rational function $\psi_1$ numerically unaltered, then the function $\varphi_1$ is rationally expressible in terms of the function $\psi_1$.**

In other words:

(117) **Any number $\omega_1$ in the domain**

$$\Omega(x_i)_1^n$$

**remaining unaltered under a subgroup $H$ of the Galoisian group $G$ is rational in**

$$\Omega(\psi_1)$$

**if $\psi_1$ is a function belonging to $H$.**

This follows from Lagrange's Theorem; or we prove it anew setting

$$G = H + Ht_2 + \ . \ . \ . + Ht_j$$

[1] The Theorem of Lagrange-Galois is also called the Theorem of Lagrange generalized.

and assuming that

$$\psi_1, \psi_2, \ . \ . \ . \ , \psi_j$$
$$\omega_1, \omega_2, \ . \ . \ . \ , \omega_j$$

are corresponding numbers. The $\omega_i$ may or may not be distinct and it is

$$\psi_1 = \psi_1(x_i)$$
$$\omega_1 = \omega_1(x_i).$$

As we did before in §62, we construct the integral function

$$\varphi(\psi)\left(\frac{\omega_1}{\psi - \psi_1} + \frac{\omega_2}{\psi - \psi_2} + \ . \ . \ . \ + \frac{\omega_j}{\psi - \psi_j}\right) = \chi(\psi),$$

where

$$\varphi(\psi) = (\psi - \psi_1)(\psi - \psi_2) \ . \ . \ . \ (\psi - \psi_j)$$

is rational. Its coefficients are rational in the $x_i$ and unaltered under the Galoisian group which cannot do more than permute the $\omega_i$ and $\psi_i$ and permute them alike. Hence the coefficients are rational numbers by property (2) of the Galoisian group, and setting

$$\psi = \psi_1$$

we find as before

$$\omega_1 = R(\psi_1).$$

This completes the adjustments which convert Lagrange's theory of general equations into Galois' theory of special equations, and recalling now the substance of Lagrange's plan we conclude for the plan of Galois:

> **Whenever the Galoisian group of a special equation has a series of subgroups each normal and of prime index in the preceding, the series beginning with the Galoisian group and ending with identity, then we can solve the general equation by algebraic operations using primitive roots of unity.**

But when these conditions do not hold, it would not seem possible to solve the special equation by algebraic operations which beside the rational operations include the extraction of roots.[1]

[1] Compare §§40 and 76. The final statement is in §82.

## §70. GENERAL EQUATION

If the complete Galoisian function $G(v)$ of an equation in $\Omega$ is reducible there, the equation is said to be **affected** in $\Omega$. We proceed to prove that

(118) **the general equation is unaffected in its domain** composed of numbers that are rational functions of its coefficients

$$c_1, c_2, \ . \ . \ . \ , c_n$$

as independent variables.

To this purpose, let first

$$x_1, x_2, \ . \ . \ . \ , x_n$$

be independent variables and

$$v_1 = \alpha_1 x_1 + \alpha_2 x_2 + \ . \ . \ . \ + \alpha_n x_n$$

a function of these variables with distinct and rational coefficients taking under the symmetric group on the $x_\iota$ the values

$$v_1, v_2, \ . \ . \ . \ , v_{n!}$$

Since the function

$$G(v) = (v - v_1)(v - v_2) \ . \ . \ . \ (v - v_{n!})$$

is symmetric in the $x_\iota$, it is rational in their elementary symmetric functions which we now replace by the independent variables $c_\iota$, and as rational function of $v$ and the $c_\iota$ we may denote it by

$$G(v|c_\iota) = (v - v_1)(v - v_2) \ . \ . \ . \ (v - v_{n!}).$$

Suppose this function is reducible in the domain of rational numbers, so that

$$G(v|c_\iota) = g(v|c_\iota) \ . \ g'(v|c_\iota) \ . \ . \ .$$

As the function identically disappears for

$$v = v_1$$

when we write $v_1$ and the $c_\iota$ in terms of the $x_\iota$, a factor

$$g(v|c_\iota)$$

must identically disappear when we do so. But with

$$g(v_1|c_\iota) = 0.$$

also

$$g(v_2|c_\iota) = 0$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$g(v_{n!}|c_\iota) = 0,$$

because permutations on the $x_\iota$ alter $v_1$ without changing the $c_i$,[1]

[1] We can apply such permutations inasmuch as $g(v|c_\iota) = 0$ is an identity in the $x_\iota$.

and $g(v|c_i)$ appears to be the same function as $G(v|c_i)$ since it has the same roots.

From this the proposition follows, because $G(v)$ is irreducible in the domain of the $c_i$ when $G(v|c_i)$ is irreducible in the domain of rational numbers.

The Galoisian group of the general equation is the symmetric group. Therefore we infer from proposition (115) that

(119) **the conjugate values which a rational function takes under the symmetric group are roots of a rational and irreducible resolvent of the general equation.**

The Galoisian group of any equation without affect is the symmetric group, and such an equation is in the Galoisian theory sometimes spoken of as a general equation.

An example of an unaffected equation other than the general is

$$x^3 - 2 = 0.$$

It has the roots

$$x_1 = 2^{1/3},\ x_2 = \omega 2^{1/3},\ x_3 = \omega^2 2^{1/3},$$

and its group is symmetric by proposition (109) since every symmetric function of its roots has a rational value while an alternating function like

$$\sqrt{\Delta} = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = 6\sqrt{-3}$$

has not.[1] Indeed we find that any rational function of its roots which has a rational value, like

$$x_1x_2 - x_3^2 = 0,$$

remains numerically unaltered under

$$S = 1,\ (12),\ (13),\ (23),\ (123),\ (132).$$

For the general equation the only rational functions of its roots which have a rational value are the symmetric functions.[2] But this is by no means true for all unaffected equations, as appears from the example.

Hence it is inaccurate to say that the plan of Lagrange breaks down whenever functions occur with numerically equal conjugate values, because this cannot happen. This would imply the

[1] Compare the explanation in §73, example (a).

[2] By the definition of its domain following proposition (118); or by proposition (119) since an assymetric function is a root of an irreducible resolvent.

existence of assymetric functions having a rational value,[1] which is not possible with the general equation. But it is true that the plan of Lagrange cannot be applied to all unaffected equations.

## §71. DUALITY OF PLANS

It is clear that the theory of Galois is precisely dual to the theory of Lagrange: in the general theory of special equations the Galoisian group takes the place which the symmetric group has in the special theory of general equations, and from there on both theories run precisely parallel.

We put together dually their principal theorems referring rationality to the domain of the equation:

| **Lagrange** | **Galois** |
|---|---|
| The group of a general equation in $x$ is the symmetric group on its roots $x_i$. | The group of a special equation in $x$ is the Galoisian group on its roots $x_i$. |
| Permutations of the symmetric group leaving a rational function $\psi_1$ of the $x_i$ formally unaltered compose a subgroup $H$ of the symmetric group. | Permutations of the Galoisian group leaving a rational function $\psi_1$ of the $x_i$ numerically unaltered compose a subgroup $H$ of the Galoisian group. |
| To every subgroup $H$ of the symmetric group belongs a rational function $\psi_1$ remaining under $H$ formally unaltered. | To every subgroup $H$ of the Galoisian group belongs a rational function $\psi_1$ remaining under $H$ numerically unaltered. |
| If a group has a subgroup $H$ of index $j$, a rational function $\psi_1$ belonging to $H$ takes $j$ formally different conjugate values $\psi_1, \psi_2, \ldots, \psi_j$ under the group which belong to subgroups conjugate with $H$ in the group. | If a group has a subgroup $H$ of index $j$, a rational function $\psi_1$ belonging to $H$ takes $j$ numerically different conjugate values $\psi_1, \psi_2, \ldots, \psi_j$ under the group which belong to subgroups conjugate with $H$ in the group. |

[1] For instance, the sum of the conjugate values which are distinct.

| Lagrange | Galois |
|---|---|
| If a group has a subgroup of index $j$, the conjugate values which a rational function belonging to $H$ takes under the group are roots of a resolvent $r(\psi) = 0$ whose degree is $j$ and whose coefficients belong to the group. | If a group has a subgroup of index $j$, the conjugate values which a rational function belonging to $H$ takes under the group are roots of a resolvent $r(\psi) = 0$ whose degree is $j$ and whose coefficients belong to the group. |
| The conjugate values which a rational function takes under the symmetric group are roots of a rational and irreducible resolvent. | The conjugate values which a rational function takes under the Galoisian group are roots of a rational and irreducible resolvent. |
| Theorem of Lagrange: If a rational function $\varphi_1$ remains formally unaltered by all those permutations of the symmetric group that leave another rational function $\psi_1$ formally unaltered, then $\varphi_1$ is rationally expressible in terms of $\psi_1$: $\varphi_1 = R(\psi_1)$. | Theorem of Lagrange-Galois: If a rational function $\varphi_1$ remains numerically unaltered by all those permutations of the Galoisian group that leave another rational function $\psi_1$, numerically unaltered, then $\varphi_1$ is rationally expressible in terms of $\psi_1$: $\varphi_1 = R(\psi_1)$. |
| If the symmetric group of a general equation is soluble, the equation is solvable by algebraic operations using primitive roots of unity. | If the Galoisian group of a special equation is soluble, the equation is solvable by algebraic operations using primitive roots of unity. |

## §72. IRREDUCIBLE EQUATION

From the fundamental properties of the Galoisian group of an equation follows that

(120) **the group of an equation $F(x) = 0$ is transitive on those, and only those, roots of the equation which are roots of an irreducible factor $f(x)$ of $F(x)$.**

For is the group intransitive on the roots $x_i$ of the function

$$F(x) = (x - x_1) \ . \ . \ . \ (x - x_k) \ . \ . \ . \ (x - x_m)$$

in $\Omega$ connecting $x_1$ only with

$$x_1, \ . \ . \ . \ , x_k$$

but no other $x_i$, the coefficients of the function

$$f(x) = (x - x_1) \ . \ . \ . \ (x - x_k)$$

remain numerically unaltered under the group and are numbers in $\Omega$ by its property (2). Hence the function $F(x)$ can be reduced in $\Omega$.

Conversely, is the function $F(x)$ reducible in $\Omega$ having there the irreducible factor $f(x)$, the coefficients of $f(x)$ are numbers in $\Omega$ and remain unaltered under the group by its property (1). Hence the group can be transitive on the roots of $f(x)$ alone.

If the equation

$$F(x) = 0$$

in $\Omega$ is reducible there and $F(x)$ has the irreducible factors

$$\begin{aligned} f_\alpha(x) &= (x - \alpha_1) \ . \ . \ . \ (x - \alpha_k) \\ f_\beta(x) &= (x - \beta_1) \ . \ . \ . \ (x - \beta_l) \\ &. \ . \ . \ . \ . \ . \ . \end{aligned}$$

in $\Omega$, so that

$$F(x) = f_\alpha(x) \cdot f_\beta(x) \ . \ . \ . \ ,$$

we conclude that the group of the equation is intransitive with the intransitive systems

$$\begin{gathered} \alpha_1, \ . \ . \ . \ , \alpha_k \\ \beta_1, \ . \ . \ . \ , \beta_l \\ . \ . \ . \ . \ . \ . \ . \end{gathered}$$

And if the group of the equation is intransitive with these systems, we conclude that the function is reducible as shown.

Hence we can note:

(121) **If an equation $F(x) = 0$ is reducible, its group $G$ is intransitive so that to every irreducible factor of $F(x)$ corresponds an intransitive system of $G$.**

And conversely:

(122) **If the group $G$ of an equation $F(x) = 0$ is intransitive, the equation is reducible so that to every intransitive system of $G$ corresponds an irreducible factor of $F(x)$.**

Further we can note:

(123) **The group of an equation is transitive when, and only when, the equation is irreducible.**

If the group $G$ of an equation is regular of degree and order $n$, also the degree of the equation is $n$. The $n$ roots of the equation then are conjugate functions under $G$, and each root belongs to a subgroup of index $n$ in $G$. This is to say that the order of the subgroup is one and the subgroup itself identity.

Since such an equation is irreducible by proposition (123) and its roots are by the Theorem of Lagrange-Galois rational functions of each other, we infer that an equation with a regular group is normal.

This condition on the group is not only necessary but also sufficient, for the roots of an equation must belong to identity to be rational functions of each other.[1] Hence

(124) **the group of a normal equation, and of such an equation alone, is regular.**

## §73. APPLICATIONS

To find the Galoisian group of a given equation is extremely difficult, but the propositions that we just proved are helpful. We know that equations may be rational in different domains, and it is clear that for different domains we may obtain different groups.

Examples:

(a) Find the group $G$ of the equation[2]

$$x^3 - 2 = 0$$

for the domain

$$\Omega = (1).$$

Since the equation is irreducible, the group $G$ is transitive and its order is a divisor of $n! = 6$ while a multiple of $n = 3$. This is to say that the order of $G$ is 6 or 3 and $G$ the symmetric or alternating group.

The discriminant of the equation is

$$\Delta = -27c_3{}^2 = -108,$$

[1] Compare §66.

[2] Compare §70.

and

$$\sqrt{\Delta} = 3c_3\sqrt{-3} = 6\sqrt{-3}.$$

Were $G$ the alternating group, the rational function

$$\sqrt{\Delta} = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

would have a rational value remaining unaltered under $G$. This is not true, whence $G$ is the symmetric group:

$$G = 1, (12), (13), (23), (123), (132).$$

---

(b) Find the group $G$ of the equation

$$x^3 - 2 = 0$$

for the domain

$$\Omega = (\omega),$$

where $\omega$ is a primitive cube root of unity.

Now

$$\sqrt{\Delta} = 6\sqrt{-3} = 6(\omega - \omega^2)$$

has a rational value and must remain unaltered under $G$. Hence $G$ contains no transposition and is the alternating group:

$$G = 1, (123), (132).$$

We observe that $G$ is regular and consequently the equation normal:

$$x_i = R(x_1),$$

which we verify remembering the roots given in §70.

---

(c) Find the group $G$ of the equation

$$x^4 - 2 = 0$$

for the domain

$$\Omega = (1).$$

Its roots are

$$x_1 = 2^{1/4}, x_2 = i2^{1/4}, x_3 = -2^{1/4}, x_4 = -i2^{1/4}.$$

The rational function

$$x_1x_3 + x_2x_4 = 0$$

has a rational value and must remain unaltered under $G$; hence we have

$$G = 1, (13), (24), (13)(24), (12)(34), (14)(23), (1234), (1432)$$

or a subgroup.

This means that the order of $G$ is a divisor of 8. Since the equation is irreducible, the order of $G$ is also a multiple of 4;

therefore it is 8 or 4. Were the order 4, then $G$ would be regular and the equation normal. But

$$x_2 \neq R(x_1),$$

whence $G$ stands as given above.

---

(d) Find the group $G$ of the equation

$$x^4 - 2 = 0$$

for the domain

$$\Omega = (i).$$

Now $G$ is regular of order 4 since the equation is irreducible and

$$x_i = R(x_1).$$

The rational functions

$$\frac{x_1}{x_4} = \frac{x_4}{x_3} = \frac{x_3}{x_2} = \frac{x_2}{x_1} = i$$

have a rational value and must remain unaltered under $G$. Hence we have

$$G = 1, (1234), (13)(24), (1432).$$

---

(e) Find the group $G$ of the equation

$$x^4 + 6x^2 + 1 = 0$$

for the domain

$$\Omega = (1).$$

Since the equation is irreducible, the group $G$ is transitive and its order a multiple of 4.

The equation is quadratic in

$$x^2 = y.$$

Therefore we have

$$y_1 y_2 = 1;$$

whence setting

$$x_1 = \sqrt{y_1},\ x_3 = -\sqrt{y_1}$$
$$x_2 = \sqrt{y_2},\ x_4 = -\sqrt{y_2}$$

we find

$$x_2 = \frac{1}{x_1},\ x_3 = -x_1,\ x_4 = -\frac{1}{x_1}.$$

These relations remain true under

$$G = 1, (12)(34), (13)(24), (14)(23).$$

We observe that $G$ is regular and consequently the equation normal, which is verified by the relations between the roots.

---

(f) Find the group $G$ of the equation

$$x^4 + 6x^2 + 1 = 0$$

for the domain

$$\Omega = (\sqrt{2}).$$

Now the equation written in the form

$$(x^2 + 3)^2 - 8 = 0$$

is reducible:

$$(x^2 + 3 + 2\sqrt{2})(x^2 + 3 - 2\sqrt{2}) = 0,$$

and the group $G$ must be intransitive connecting the roots of the factors only.

From the relations

$$x_2 = \frac{1}{x_1}, \; x_3 = -x_1, \; x_4 = -\frac{1}{x_1}$$

follows

$$x_1x_2 = 1, \; x_1x_4 = -1,$$

whence one factor must have the roots $x_1$ and $x_3$ and the other factor the roots $x_2$ and $x_4$. We set

$$x_1x_3 = 3 \pm 2\sqrt{2}, \; x_2x_4 = 3 \mp 2\sqrt{2};$$

the group leaving these rational functions with rational values numerically unaltered and connecting the roots of the factors is

$$G = 1, (13)(24).$$

## §74. IMPRIMITIVE EQUATION

Suppose that the equation

$$f(x) = 0$$

of degree $n$ in $\Omega$ is irreducible there, so that its Galoisian group is transitive by proposition (123), and set

$$x_1 = \alpha_1.$$

If the algebraic domain $\Omega(\alpha_1)$ is imprimitive, it contains an imprimitive number $\theta_1$ equal to some of its conjugate values:

$$\theta_1 = \rho(\alpha_1) = \ldots = \rho(\alpha_m),$$

and taking under the transitive group of the equation the other conjugate values

$$\theta_2 = \rho(\beta_1) = \ . \ . \ . \ = \rho(\beta_m)$$

$$. \quad . \quad . \quad . \quad . \quad . \quad .$$

This divides the roots of the equation into the systems

$$\alpha_1, \ . \ . \ . \ , \alpha_m$$
$$\beta_1, \ . \ . \ . \ , \beta_m$$
$$. \quad . \quad . \quad . \quad . \quad . \quad .$$

such that no two roots in different systems are alike.[1]

Since by proposition (107) the relation

$$\rho(\alpha_i) = \rho(\alpha_j)$$

must remain true when operated on by the permutations of the Galoisian group, it follows that those permutations interchange either roots within systems or else entire systems. Hence

(125) **an equation with an imprimitive algebraic domain has an imprimitive group;**

such an equation is called an **imprimitive equation.**[2]

The conjugate values

$$\theta_1, \theta_2, \ . \ . \ . \ , \theta_k$$

are roots of the function

$$\varphi(\theta) = (\theta - \theta_1)(\theta - \theta_2) \ . \ . \ . \ (\theta - \theta_k)$$

which is rational in $\Omega$ by property (2) of the Galoisian group, the group not more than permuting the $\theta_i$, and is irreducible there by proposition (115).

Denoting by

$$\omega_1 = S_1(\alpha_i)$$
$$\omega_2 = S_2(\beta_i)$$
$$. \quad . \quad . \quad . \quad . \quad . \quad .$$

symmetric functions on the imprimitive systems, we construct the integral function

$$\varphi(\theta)\left(\frac{\omega_1}{\theta - \theta_1} + \frac{\omega_2}{\theta - \theta_2} + \ . \ . \ . \ + \frac{\omega_k}{\theta - \theta_k}\right) = \chi(\theta)$$

[1] The $\beta_i$ are all unlike, with the $\alpha_i$; and none equals an $\alpha_i$ as from

$$\alpha_i = \beta_j$$

would follow

$$\rho(\alpha_i) = \rho(\beta_j)$$

and two entire systems were indentical.

[2] It can be proved that an equation with a primitive algebraic domain has a primitive group.

whose coefficients, rational in $\Omega$ and unaltered under the Galoisian group which cannot do more than permute the $\omega_i$ and $\theta_i$ alike, are numbers in $\Omega$ by property (2) of the group. Setting

$$\theta = \theta_1,$$

we find

$$\omega_1 = R(\theta_1),$$

this relation being rational in $\Omega$.

It follows that the function[1]

$$f_\alpha(x \mid \theta_1) = (x - \alpha_1) \ . \ . \ . \ (x - \alpha_m)$$

is rational in $\Omega$; it is also irreducible there,[2] since those permutations of the Galoisian group which leave $\theta_1$ unaltered are transitive on the $\alpha_i$. This means that the function

$$f_\alpha(x) = (x - \alpha_1) \ . \ . \ . \ (x - \alpha_m)$$

is rational and irreducible in $\Omega(\theta_1)$.

As the degree of an irreducible equation is also the degree of its algebraic domain, we have the proposition:

(126) **An imprimitive domain** $\Omega(\alpha_1)$ **of degree** $n$
**on** $\Omega$ **is identical with the domain** $\Omega'(\alpha_1)$ **of degree** $m$
**on** $\Omega$ **if** $\Omega' = \Omega(\theta_1)$ **is of degree** $k$
**on** $\Omega$ **and** $k \, . \, m = n.$

For the example (d) of §73 we have:

$$\begin{array}{c|rcl|c} & \Omega(2^{1/4}) & = & \Omega'(2^{1/4}) & \\ 4 & \Omega(2^{1/2}) & = & \Omega' & 2 \\ & \Omega & = & (i). & \end{array}$$

Here

$$\theta_1 = (2^{1/4})^2 = (-2^{1/4})^2$$
$$\theta_2 = (i2^{1/4})^2 = (-i2^{1/4})^2,$$

whence

$$\alpha_1 = 2^{1/4}, \quad \alpha_2 = -2^{1/4}$$
$$\beta_1 = i2^{1/4}, \quad \beta_2 = -i2^{1/4}$$
$$\theta_1 = 2^{1/2}, \quad \theta_2 = -2^{1/2}.$$

## §75. REDUCTION OF GROUP

Any equation which is rational in a domain $\Omega$ is with stronger reason rational in a domain on $\Omega$ obtained by adjunction. The

[1] It is a function also of $\theta_1$.

[2] By proposition (123).

group of the equation may be different for the new domain, and we prove the proposition:

(127) **The group of an equation is reduced to a subgroup when we adjoin to the domain of the equation a rational function of its roots which belongs to the subgroup.**

Let the group G of the equation

$$F(x) = a_0x^n + a_1x^{n-1} + \ . \ . \ . + a_n = 0$$

rational in the domain

$$\Omega = (a_i)$$

of its coefficients, or any wider domain $\Omega$, have a subgroup $H$ and let

$$\psi_1 = \psi_1(x_i)$$

be a function rational in $\Omega$ which belongs to $H$ and adjoined to $\Omega$ produces the domain

$$\Omega' = (a_i, \psi_1).$$

Since any function $\chi$ of the $x_i$ rational in $\Omega'$ is a function of $\psi_1$ and the $x_i$ rational in $\Omega$—which we may express by setting

$$\chi(x_i) = \chi(x_i|\psi_1),$$

it also is a function of the $x_i$ rational in $\Omega$ because $\psi_1$ is.

Assuming first that such a function $\chi$ has a value $\rho$ ration-

$$\Omega' = \Omega(\psi_1),$$

so that

$$\rho = \rho(\psi_1)$$

is a function[1] rational in $\Omega$, we have a relation

$$\chi(x_i|\psi_1) = \rho(\psi_1)$$

between the $x_i$ which is rational in $\Omega$. This relation remains true under $G$ by proposition (107), while the function $\rho(\psi_1)$ is unaltered by the subgroup $H$ of $G$. Hence we infer that also the function $\chi(x_i)$ is unaltered by $H$.

Assuming second that the function $\chi$ remains unaltered by the subgroup $H$ of $\psi_1$, we infer from the Theorem of Lagrange-Galois that it has a value rational in $\Omega'$, whence it appears that $H$ possesses the fundamental properties of the Galoisian group in $\Omega'$, and the proposition follows.

[1] Cf. proposition (91).

It implies that

(128) **a normal domain $\Omega(x_i)_1^n$ of degree $r_g$ on $\Omega$ is identical with the domain $\Omega'(x_i)_1^n$ of degree $r_h$ on $\Omega$ when $\Omega' = \Omega(\psi_1)$ is of degree $j$ on $\Omega$ and $j = r_g/r_h$,**

in the notation of §17. For the group of an equation in its normal domain is identity. If its group in $\Omega^*$ is $H^*$, the degree of the resolvent in $\Omega^*$ for identity equals the index of identity in $H^*$, which is the order of $H^*$, and the degree of this resolvent is also the degree of the normal domain $\Omega^*(x_i)$ on $\Omega^*$.

It is clear that to solve the equation

$$F(x) = 0$$

means to widen its domain by adjunction so that its roots $x_i$ are rationally known and its group $G$ is reduced to identity, therefore means to obtain its normal domain

$$\Omega(v_1) = \Omega(x_i).$$

But we know now that it is not necessary to adjoin at once an irrationality which is primitive there: we may widen the domain $\Omega$ of the equation by successive adjunctions reducing its group $G$ from subgroup to subgroup.

Let in the domain $\Omega$ of the equation its group be

$$G = 1, \ldots, t$$

and its resolvent

$$g(v) = (v - v_1) \ldots (v - v_t) = 0,$$

let in the domain $\Omega'$ its group be

$$H = 1, \ldots, s$$

and its resolvent

$$h(v) = (v - v_1) \ldots (v - v_s) = 0.$$

The function $h(v)$ is rational in $\Omega'$ but may be expressed as

$$h(v|\psi_1) = (v - v_1) \ldots (v - v_s)$$

rational in $\Omega$. Since the permutations of $H$ are contained in $G$, $h(v)$ is a factor of $g(v)$ and $g(v)$ appears to be reducible in $\Omega'$.

Under the group

$$G = H + \ldots + Ht$$

of the equation the function $\psi_1$ takes $j$ conjugate values

$$\psi_1, \ldots, \psi_t;$$

and the function $h(v|\psi_1)$ goes into

$$h(v|\psi_1), \ldots, h(v|\psi_t)$$

such that

$$h(v|\psi_1) = (v - v_1) \ldots (v - v_s)$$
$$\cdots\cdots\cdots$$
$$h(v|\psi_t) = (v - v_t) \ldots (v - v_{st})$$

and the unlike[1] sets

$$v_1, \ldots, v_s$$
$$\cdots\cdots\cdots$$
$$v_t, \ldots, v_{st}$$

contain all values of $v_1$ conjugate under $G$. Hence we can say that

(129) **adjunction reducing the Galoisian group splits the Galoisian resolvent into conjugate factors:**

$$\boxed{g(v) = h(v|\psi_1) \ldots h(v|\psi_t)};$$

these factors are rational in

$$\Omega(\psi_1), \ldots, \Omega(\psi_t)$$

and of the same degree $r_h$ in $v$.

If the subgroup $H$ is normal in $G$, the domain $\Omega(\psi_1)$ is normal on $\Omega$ since we have

$$\psi_t = R(\psi_1)$$

for any $\psi_t$ by the Theorem of Lagrange-Galois; and the Galoisian resolvent under $G$ splits by adjunction of $\psi_1$ into rational factors each yielding the group $H$.

If $H$ is not normal in $G$, a normal domain on $\Omega$ is produced[2] when we adjoin the function

$$\theta_1 = \alpha_1\psi_1 + \ldots + \alpha_t\psi_t,$$

the $\alpha_i$ chosen such that $\theta_1$ is altered by every permutation between the $\psi_i$. The adjunction of $\theta_1$ reduces $G$ by proposition (127) to the greatest common subgroup $D$ of

$$H, \ldots, H_t$$

and is equivalent to the adjunction of all $\psi_i$ by proposition (102).

We have come to another point of vantage from which we can survey the plan of Galois. The essence of solving an equation

[1] Since $v_1$ takes different values under different permutations of $G$.

[2] By propositon (101).

is this: by adjoining proper irrationalities we reduce the Galoisian group of the equation and split its Galoisian resolvent.

With this in mind, we turn to resolvents which solved give proper irrationalities. But in doing so we call attention to the circumstance that it will be important not to confuse the Galoisian resolvent $g(v) = 0$ and the resolvent $r(\psi) = 0$ of Lagrange's conception, the ordinary resolvent so to say. The Galoisian resolvent illumines the solution, but the ordinary resolvent is the one that gives it.

The function $\psi_1$ is a root of the resolvent equation $r(\psi) = 0$ and

(130) **the group of the resolvent equation**

$$\boxed{r(\psi) = 0}.$$

**is in the domain of the given equation**

$$\boxed{F(x) = 0}$$

**the group $\Gamma$ of permutations which the $\psi_i$ undergo when the $x_i$ are operated on by the permutations of $G$.**

For any function of the $\psi_i$ rational in the domain $\Omega$ of the given equation is also a function rational there of the $x_i$:

$$\Phi(\psi_i) = \varphi(x_i),$$

and if $\Phi$ remains unaltered under $\Gamma$, then $\varphi$ remains unaltered under $G$, and conversely.

Therefore $\Phi$ remains unaltered under $\Gamma$ if it has a value rational in $\Omega$, for $\varphi$ then remains unaltered under $G$ by property (1) of $G$; and $\Phi$ has a value rational in $\Omega$ if it remains unaltered under $\Gamma$, for $\varphi$ then has such a value by property (2) of $G$. Hence the proposition follows because $\Gamma$ has the fundamental properties of the group in $\Omega$.

We note that the group $\Gamma$ of the resolvent is transitive since the resolvent is irreducible.[1]

The group $\Gamma$ is abstractly defined by the factor-group

$$G/H = \Gamma$$

when $H$ is normal in $G$, else by the factor-group

$$G/D = \Gamma.$$

[1] By propositions (115) and (123).

Consequently the resolvent in $\theta$ has the same group as the resolvent in $\psi$; it is alike in solution although of higher degree when $H$ is not normal.

If the $\psi_i$ belong to subgroups of $G$ having no permutation but identity in common, then

$$D = 1,$$

abstractly

$$\Gamma = G,$$

and the resolvent

$$r(\psi) = 0$$

is called a **total resolvent.** The adjunction of its roots reduces the Galoisian group to identity and splits the Galoisian resolvent into linear factors. This implies the solution of the given equation, yet nothing is gained because the group of a total resolvent is identical with that of the given equation and its solution equally difficult.

We observe that a total resolvent is Galoisian in the wider sense of this term when the $\psi_i$ belong to identity.

If the groups of the $\psi_i$ have more permutations than identity in common, then the resolvent

$$r(\psi) = 0$$

is called a **partial resolvent.** The adjunction of its roots reduces the Galoisian group to a normal subgroup $D$ and splits the Galoisian resolvent into conjugate factors.

Suppose the group $G$ of the given equation has the composition-series

$$G \quad N \quad J \quad \ldots \quad 1.$$

The smallest groups for partial resolvents are defined by the abstract quotients

$$G/N,\ N/J,\ \ldots$$

of the series, and it is the purpose of Galois to replace the solution of the given equation by a successive solution of partial resolvents with such groups. Their roots adjoined to the domain of the given equation reduce its group along the series of composition.

Such partial resolvents we may call **resolvents of composition,** and we infer from proposition (124) that

(131) **for soluble groups resolvents of composition are normal.**

For the group $\Gamma$ of any such resolvent is circular, since it is of prime degree and order:[1]

$$\Gamma = \sigma, \sigma^2, \ . \ . \ . \ , \sigma^p, \qquad [\sigma^p = 1$$

where

$$\sigma = (\psi_1 \psi_2 \ . \ . \ . \ \psi_p)$$

and $p$ is a factor of composition.

For soluble groups resolvents of composition are even binomal, we now recall,[2] and can be solved by algebraic operations on rational numbers and primitive roots of unity if constructed in the form

$$r(\epsilon, \psi) = 0,$$

where $(\epsilon, \psi)$ is Lagrange's solvent.

## §76. NATURAL IRRATIONALITY

To solve an equation

$$F(x) = a_0x^n + a_1x^{n-1} + \ . \ . \ . \ + a_n = 0$$

rational in the domain

$$\Omega = (a_\iota)$$

of its coefficients, or any larger domain $\Omega$, we must widen its domain to the normal domain

$$\Omega(v_1) = \Omega(x_i)$$

where its roots are rational, its Galoisian group reduced to identity, its Galoisian resolvent split into linear factors.

We achieve this by successive adjunction of numbers contained in the normal domain; such numbers are roots of resolvent equations and called **natural irrationalities** after Kronecker.

Thus we know exactly what we need to solve the equation. Can we solve it? We recall that a sufficient condition is that the group of the equation be soluble. Is it necessary? When this condition holds we can construct and solve resolvents, otherwise we can not. And with a last glimpse of hope we now ask the question: When natural irrationalities cannot be computed, are there perhaps other irrationalities which can solve the equation?

[1] By proposition (25), and a circular group of same degree as the equation is regular by proposition (59).

[2] Proposition (55).

A negative answer is given with the proposition that

(132) **every possible reduction of the Galoisian group is effected by adjunction of natural irrationalities.**

Suppose that adjoining an irrational number $y$ we produce a domain

$$\Omega_y = (a_i, y)$$

in which the Galoisian resolvent is

$$h(v|y) = (v - v_1) \; . \; . \; . \; (v - v_s) = 0.$$

We proved for any domain that the permutations yielded by the resolvent compose a unique group

$$H = 1, \; . \; . \; . \; , s.$$

If now $h(v|y)$ is a factor of $g(v)$, then $H$ is a subgroup of $G$ and we reduce the Galoisian group by adjoining $y$. But the same reduction, of course, results when we adjoin any function $\psi$ of the $x_i$ belonging to $H$. And furthermore the domain

$$\Omega_\psi = (a_i, \psi)$$

is contained in the domain

$$\Omega_y = (a_i, y),$$

since the coefficients of $h(v|y)$ are rational in no smaller domain on $\Omega$ than $\Omega_\psi$ and also are rational in $\Omega_y$.[1] It appears that we can certainly construct $\psi$ if we can construct $y$, and that no solution of the equation is possible which cannot be effected also by natural irrationalities.

Considering that any complete solution of the equation reduces its group to identity, we now conclude for all equations, since the theory of Galois includes that of Lagrange:

> **Whenever the Galoisian group of an equation has a series of subgroups each normal and of prime index in the preceding, the series beginning with the Galoisian group and ending with identity, then we can solve the equation by algebraic operations on rational numbers and primitive roots of unity. But when these conditions do not hold, then it is not possible to solve the equation by algebraic operations on such numbers.**[2]

[1] For the coefficients belong to $H$, and so does $\psi$; moreover $\psi$ is a root of an irreducible resolvent and primitive in $\Omega_\psi$.

[2] Compare §§40 and 69. The final statement is in §82. For primitive roots of unity compare footnote to §40.

# CHAPTER XII

## SPECIAL EQUATIONS

### §77. ABELIAN EQUATION

As we did in the preceding chapter, we shall assume here that the equations we treat of have no double roots. Furthermore it will be observed that the typical equations we treat of are irreducible.

An irreducible equation whose group is Abelian we call an **Abelian equation.**

Suppose that

$$f(x) = 0$$

is an Abelian equation with $n$ roots $x_i$. Then its group $G$ is transitive by proposition (123) and contains a permutation $s_i$ converting $x_1$ into $x_i$.

Those permutations of $G$ that leave $x_1$ unaltered compose a subgroup $H$, and those that leave $x_i$ unaltered compose a subgroup

$$H_i = s_i^{-1}Hs_i$$

conjugate with $H$. But subgroups of an Abelian group are normal, by proposition (32),[1] and we have

$$H_i = H = 1,$$

for identity alone leaves every $x_i$ unaltered.

If a permutation $s_i'$ other than $s_i$ would convert $x_1$ into $x_i$, then the permutation $s_i's_i^{-1}$ would leave $x_1$ unaltered, so that

$$s_i's_i^{-1} = 1$$

and

$$s_i' = s_i.$$

Hence the group of the equation is

$$G = 1, s_2, \ldots, s_n.$$

The group is regular, and it follows by proposition (124) that

(133) **every Abelian equation is normal.**

[1] Cf. also §56.

This means that all roots $x_i$ are rational functions of $x_1$, and we can set

$$x_1 = R_1(x_1),\ x_2 = R_2(x_1),\ \ldots,\ x_n = R_n(x_1).$$

By proposition (107), any relations

$$x_i = R_i(x_1),\ x_k = R_k(x_1)$$

remain true when we apply the permutation $s_k$ of the group to one and the permutation $s_i$ of the group to the other:

$$x_{ik} = R_i(x_k),\ x_{ki} = R_k(x_i).$$

But with

$$s_i s_k = s_k s_i,$$

which is true for an Abelian group, also

$$x_{ik} = x_{ki}$$

and

$$R_i(x_k) = R_k(x_i).$$

Hence

(134) **for an Abelian equation in $x$ the relation holds:**

$$\boxed{R_i[R_k(x_1)] = R_k[R_i(x_1)]}.$$

Conversely,

(135) **when this relation holds for the roots $x_i$ of an equation while any**

$$\boxed{x_i = R_i(\ _1)},$$

**the group of the equation is Abelian.**

For it follows from

$$x_{ik} = R_i(x_k),\ x_{ki} = R_k(x_i)$$

that

$$x_{ik} = x_{ki}$$

and

$$s_i s_k = s_k s_i,$$

which proves the proposition.

It is to be noted that in proving the converse we did not qualify the equation as irreducible.

## §78. CYCLIC EQUATION

An irreducible equation whose group is cyclic we call a **cyclic equation.**

Suppose that

$$f(x) = 0$$

is a cyclic equation with $n$ roots $x_i$. Its group is cyclic by definition and transitive by proposition (123), therefore by proposition (59) circular:

$$G = \{s\},$$

containing all powers of the circular permutation

$$s = (x_1 x_2 \ . \ . \ . \ x_n).$$

It follows from propositions (25) and (124) that

(136) **every normal equation of prime degree is cyclic;**
in particular resolvents of composition for soluble groups are cyclic equations by proposition (131).

Since cyclic groups are Abelian, cyclic equations are also; consequently cyclic equations are normal by proposition (133).

Whether the group of the equation

$$f(x) = 0$$

is $G$ or a subgroup

$$H = \{s^k\}$$

of $G$, the integral function

$$f(x)\left(\frac{x_2}{x - x_1} + \frac{x_3}{x - x_2} + \ . \ . \ . \ + \frac{x_1}{x - x_n}\right) = \chi(x)$$

has coefficients unaltered by the group and rational by its property (2). Setting

$$x = x_i$$

and

$$\frac{\chi(x_i)}{f'(x_i)} = R(x_i),$$

we find[1] that

$$x_2 = R(x_1),\ x_3 = R(x_2),\ . \ . \ . \ ,\ x_n = R(x_{n-1}),\ x_1 = R(x_n).$$

If we put

$$R[R(x_1)] = R^2(x_1),\ R[R^2(x_1)] = R^3(x_1),\ . \ . \ .$$

where exponents do not denote powers, we have the relations

$$x_2 = R(x_1),\ x_3 = R^2(x_1),\ . \ . \ . \ ,\ x_n = R^{n-1}(x_1),\ x_1 = R^n(x_1),$$

or

$$x_{z+1} = R^z(x_1)$$

with

$$z \equiv 1, 2, \ . \ . \ . \ (\text{mod } n),$$

[1] Compare the procedure in §62.

and say that

(137) **the roots of an equation compose a cycle when the group of the equation is circular of same degree as the equation or a subgroup of the circular group.**

In case of a cyclic equation we obtain the cyclic relation of its roots more simply by applying the permutations of $G$ to the relation

$$x_2 = R(x_1),$$

which is true for any normal equation.

Conversely,

(138) **when the roots of an equation compose a cycle, the group of the equation is circular of same degree as the equation or a subgroup of the circular group.**

For let any permutation of the group be

$$t = \begin{pmatrix} x_1 & \cdots & x_{z+1} & \cdots \\ x_\alpha & \cdots & x_{\zeta+1} & \cdots \end{pmatrix}.$$

Applied to the cyclic relation[1] it gives

$$x_{\zeta+1} = R^z(x_\alpha)$$

by proposition (107). But

$$R^z(x_\alpha) = R^z[R^\alpha(x_1)] = R^{z+\alpha}(x_1) = x_{z+\alpha},$$

whence

$$x_{\zeta+1} = x_{z+\alpha}$$

and

$$\zeta + 1 \equiv z + \alpha \pmod{n}.$$

It appears that

$$t = \begin{pmatrix} 1 & 2 & 3 & \cdots \\ \alpha & \alpha+1 & \alpha+2 & \cdots \end{pmatrix}.$$

With the modulus $n$ understood it is

$$t = \begin{pmatrix} z+1 \\ z+\alpha \end{pmatrix};$$

but this is the $(\alpha - 1)$st power of the circular permutation[2]

$$s = \begin{pmatrix} z \\ z+1 \end{pmatrix} = (12 \ldots n).$$

Hence the group of the equation is $G$ or a subgroup $H$, and the proposition follows. If the group of the equation is $G$, the equation is cyclic.

[1] $x_{z+1} = R^z(x_1)$.

[2] Cf. §58.

## §79. ROOTS OF UNITY

To solve an equation by algebraic operations, we need prime roots of unity which are primitive, as we recall.

Roots of unity are defined by an equation

$$x^n - 1 = 0;$$

there are $n$ distinct roots since the discriminant[1] of this equation cannot vanish. Such roots whose $n$-th powers, and no lower ones, equal unity are called **primitive n-th roots of unity.**

If $\epsilon$ is some root of this equation, its powers are also, for

$$(\epsilon^\alpha)^n = (\epsilon^n)^\alpha = 1.$$

And if $m$ is its lowest power that equals unity, we say that $\epsilon$ belongs to the exponent $m$ and find that

$$\epsilon, \epsilon^2, \ldots, \epsilon^m$$

are all unlike. For were

$$\epsilon^\alpha = \epsilon^\beta,$$

such that

$$\alpha > \beta,$$

from

$$\epsilon^{\alpha-\beta} = 1$$

would follow that a power of $\epsilon$ lower than $m$ is unity.

Since

$$\epsilon^{\alpha+m} = \epsilon^\alpha,$$

we can arrange the $n$ roots

$$\begin{array}{llll} \epsilon, & \epsilon^2, & \ldots, & \epsilon^m \\ \epsilon^{m+1}, & \epsilon^{m+2}, & \ldots, & \epsilon^{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \epsilon^{km+1}, & \epsilon^{km+2}, & \ldots, & \epsilon^n \end{array}$$

of the equation so that roots in the same column have the same value. It then appears that $m$ is a divisor of $n$ since

$$(k+1)m = n.$$

Whenever

$$n = p$$

where $p$ is prime, also

$$m = p$$

while

$$k = 0,$$

[1] Cf. §27.

and $\epsilon$ is a primitive root of unity whose powers

$$\epsilon, \ \epsilon^2, \ \ldots, \ \epsilon^p$$

are all distinct and identical with

$$\epsilon^\alpha, \epsilon^{2\alpha}, \ \ldots, \ \epsilon^{p\alpha}$$

except for the sequence.

Hence it appears that

(139) **a prime root of unity other than 1 is primitive.**

The equation

$$c(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \ldots + x + 1 = 0$$

whose root it is we call a **cyclotomic equation.**

But before we can approach this equation, we must digress into the theory of numbers.

## §80. CONGRUENCE

Let $\varphi(n)$ denote the number of positive integers smaller than $n$ which are prime to $n$.

When

$$n = p$$

is a prime number, then

$$1, 2, \ldots, p - 1$$

all are prime to $p$ and

$$\varphi(p) = p - 1.$$

When

$$n = p^\alpha,$$

we eliminate from

$$0, 1, 2, \ldots, n - 1$$

the $n/p$ numbers

$$0, p, 2p, \ldots, \left(\frac{n}{p} - 1\right)p,$$

leaving those prime to $n$, and find

$$\varphi(n) = n - \frac{n}{p} = n\left(1 - \frac{1}{p}\right).$$

When

$$n = ab$$

where $a$ and $b$ are one prime to the other, we put

$$z = ay - bx$$

and let

$$x = 0, 1, 2, \ldots, a - 1$$
$$y = 0, 1, 2, \ldots, b - 1.$$

This gives $n$ values for $z$ which are all distinct modulo $n$. For assuming that

$$z \equiv z'(\text{mod } n),$$

we have[1]

$$z - z' = a(y - y') - b(x - x')$$

divisible by $n$ and consequently $b(x - x')$ by $a$. But $b$ is prime to $a$, hence $x - x'$ is divisible by $a$ and we have

$$x - x' = 0,$$

as it is smaller than $a$. It appears that

$$x = x',$$

and similarly that

$$y = y'.$$

Thus

$$z \equiv 0, 1, 2, \ldots, n - 1 \ (\text{mod } n).$$

To eliminate those values of $z$ that are not prime to $n$, we must eliminate those values of $x$ that are not prime to $a$ and those values of $y$ that are not prime to $b$, whence

$$\varphi(n) = \varphi(a)\varphi(b).$$

When

$$a = p^\alpha$$

and

$$b = q^\beta$$

where $p$ and $q$ are prime, then $a$ and $b$ are one prime to the other, and we have the proposition:

(140) **If there are $\varphi(n)$ positive integers smaller than $n$ which are prime to $n$ and**

$$\boxed{n = p^\alpha q^\beta},$$

**then**

$$\boxed{\varphi(n) = n\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)}.$$

Among the values of $z$ one must be

$$z = c' + kn$$

where $c'$ is an assigned number. Hence the equation

$$ay - bx = c' + kn$$

[1] $z' = ay' - bx'$.

has one solution; and so has any **Diophantine equation**

$$ay - bx = c$$

where $a$ and $b$ are one prime to the other.

It follows that $bx + c$ is divisible by $a$ and the linear congruence

$$bx + c \equiv 0 (\mathrm{mod}\, a)$$

has a solution, and only one. This is to say that there is one value of $x$ satisfying the congruence and at the same time determining $y$; it is said to be a **root** of the congruence.

Likewise, for any integral function

$$f(x) = a_0x^n + a_1x^{n-1} + \ . \ . \ . + a_n$$

with integral coefficients and $a_0$ not divisible by $p$ the congruence

$$f(x) \equiv 0 (\mathrm{mod}\, p)$$

is said to have a root $\alpha$ if $f(\alpha)$ is divisible by $p$.

Using such a function of degree $n$, we prove that

(141) **a congruence of degree $n$ modulo $p$ where $p$ is prime has no more than $n$ roots.**

This is true for

$$n = 1,$$

and the proposition follows by complete induction.

Assuming that $x$ and $\alpha$ are indeterminate, we can set[1]

$$f(x) = (x - \alpha)g(x) + f(\alpha),$$

where $g(x)$ has the same qualifications as $f(x)$ but is of degree $n - 1$. From this relation we conclude that $(x - \alpha)g(x)$ is divisible by $p$ if $f(x)$ and $f(\alpha)$ are, so that $g(x)$ then is divisible by $p$ if $x$ is not congruent to $\alpha$.

If now the proposition is true for $n - 1$, then $g(x)$ has no more than $n - 1$ incongruent roots and consequently $f(x)$ no more than $n$ such roots, whence the proposition is true also for $n$. The additional root must be $\alpha$ making both terms on the right divisible by $p$ although $g(\alpha)$ is not divisible.

But differing from equations, a congruence may have less than $n$ roots; and even no roots at all like

$$x^2 + 1 \equiv 0 (\mathrm{mod}\ 3).$$

Yet there exists a peculiar congruence that cannot have less than $n$ roots, as we shall see presently.

[1] Cf. §5.

## §81. FERMAT'S THEOREM

According to **Fermat's Theorem,**[1]

(142) **for a prime number $p$ one has**

$$\boxed{a^{p-1} \equiv 1 \pmod{p}}$$

**if $a$ is a number not divisible by $p$.**

This is true because it can be proved by complete induction that

$$a^p \equiv a(\text{mod } p).$$

For

$$a = 1$$

evidently

$$1^p \equiv 1(\text{mod } p).$$

For

$$a = 2$$

we have

$$2^p = (1 + 1)^p = 1 + p + \frac{p(p-1)}{2} + \ . \ . \ . + p + 1$$
$$\equiv 2(\text{mod } p)$$

since all middle terms are divisible by $p$. And if it is true that

$$a^p \equiv a(\text{mod } p),$$

then also

$$(a + 1)^p = a^p + pa^{p-1} + \frac{p(p-1)}{2} a^{p-2} + \ . \ . \ . + pa + 1$$
$$\equiv a + 1(\text{mod } p).$$

It appears that the congruence

$$x^p \equiv x(\text{mod } p)$$

has $p$ roots

$$0, 1, 2, \ . \ . \ . \ , p - 1,$$

and the congruence

$$x^{p-1} \equiv 1(\text{mod } p)$$

has $p - 1$ roots

$$1, 2, \ . \ . \ . \ , p - 1.$$

The two congruences are essentially alike since the first has only an additional root 0. These are the peculiar congruences with a number of roots equal to their degree.

If $m$ is the smallest positive number for which

$$a^m \equiv 1(\text{mod } p),$$

[1] Fermat lived 1601–1665.

then $a$ is said to belong to the exponent $m$ modulo $p$, and if also

$$a^n \equiv 1 (\text{mod } p),$$

then $n$ is a multiple of $m$, so that $m$ always is a divisor of $p - 1$.

A number $g$ that belongs to the exponent $p - 1$ is called a **primitive root of the prime number $p$.**

For the prime number $p = 2$ it is readily seen that $g = 1$. If $p$ is an odd prime,[1] we set

$$p - 1 = a^\alpha b^\beta c^\gamma \ldots$$

where $a, b, c$ are prime numbers and prove that

(143) **there is a number $A$ belonging to the exponent $a^\alpha$.**

For the congruence

$$x^{\frac{p-1}{a}} \equiv 1 (\text{mod } p)$$

is of lower degree than $p - 1$ and has by proposition (141) fewer than $p - 1$ roots. Hence there is a number $y$ among

$$1, 2, \ldots, p - 1$$

which does not satisfy the congruence. We set

$$A = y^{b^\beta c^\gamma} \cdots$$

and have

$$A^{a^\alpha} \equiv 1 (\text{mod } p).$$

If now $A$ belongs to the exponent $m$, then $m$ is a divisor of $a^\alpha$ and consequently a power of $a$. It must be equal to $a^\alpha$, for were

$$m < a^\alpha,$$

we should have[2]

$$A^{a^{\alpha-1}} \equiv 1$$

and

$$y^{a^{\alpha-1} b^\beta c^\gamma \cdots} = y^{\frac{p-1}{a}} \equiv 1 (\text{mod } p),$$

which is a contradiction.

Likewise it can be shown that there are numbers $B, C, \ldots$ belonging to the exponents $b^\beta, c^\gamma, \ldots$ respectively.

We now prove that

(144) **the product $g = A\ B\ C \ldots$ belongs to the exponent $p - 1$.**

It cannot belong to an exponent $h$ such that

$$h < p - 1,$$

[1] An odd prime is any prime number other than 2.

[2] $A^m$ to some power $a^k$ would give $A^{a^{\alpha-1}}$.

for that gives[1]

$$\frac{p-1}{h} = j$$

where $j$ is a positive integer greater than 1 and hence divisible by some prime number $a, b, c, \ldots$ Suppose it is divisible by $a$; then it follows from

$$g^h \equiv 1$$

that

$$g^{\frac{p-1}{a}} \equiv 1 (\text{mod } p)$$

since $(p-1)/a$ is divisible by $h$ if $j$ is divisible by $a$. In other words we then have

$$A^{\frac{p-1}{a}} B^{\frac{p-1}{a}} C^{\frac{p-1}{a}} \ldots \equiv 1.$$

But $(p-1)/a$ is divisible by $b^\beta, c^\gamma, \ldots$ Hence

$$B^{\frac{p-1}{a}} \equiv 1, C^{\frac{p-1}{a}} \equiv 1$$

leaving

$$A^{\frac{p-1}{a}} \equiv 1 (\text{mod } p).$$

This is not true since $(p-1)/a$ is not divisible by $a^\alpha$, and the proposition follows.[2]

It appears that $g$ is a primitive root of the prime number $p$. Its powers

$$1, g, g^2, \ldots, g^{p-2}$$

are identical with

$$1, 2, 3, \ldots, p-1$$

except for the sequence, since they are $p - 1$ incongruent numbers not divisible by $p$.

If $p - 1$ and $k$ have the greatest common factor $f$ and

$$p - 1 = fQ$$
$$k = fq,$$

where

$$q < Q$$

and prime to it, then

$$g^{kl} = g^{fql} \equiv 1$$

[1] $g^{p-1} = (ABC \ldots)^{p-1} \equiv 1$ by proposition (143).

[2] This proof was given by the great Gauss; he lived 1777–1855.

only when $l$ is divisible by $Q$. Hence $g^k$ belongs to the exponent $Q$, and as $q$ can have $\varphi(Q)$ values, there are $\varphi(Q)$ incongruent numbers belonging to the exponent $Q$.

For

$$Q = p - 1$$

we find that

(145) **there are $\varphi(p - 1)$ primitive roots of $p$.**

## §82. CYCLOTOMIC EQUATION

Returning to the cyclotomic equation for prime roots of unity, we prove for odd primes that[1]

(146) **a cyclotomic equation for prime roots of unity is irreducible in the domain of rational numbers.**

In

$$c(x) = \frac{x^p - 1}{x - 1}$$

we set

$$x = y + 1.$$

Then

$$c(y + 1) = y^{p-1} + py^{p-2} + \frac{p(p-1)}{2}y^{p-3} + \cdots + \frac{p(p-1)}{2}y + p.$$

The right member is irreducible by proposition (9), and with $c(y + 1)$ also $c(x)$ is irreducible.

Hence the group $G$ of the cyclotomic equation

$$c(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 = 0$$

is transitive[2] and its order a multiple[3] of $p - 1$.

The roots of this equation are

$$x_1 = \epsilon,\ x_2 = \epsilon^2,\ \ldots,\ x_{p-1} = \epsilon^{p-1}, \qquad [\epsilon^p = 1$$

so that

$$x_i = x_1^i,$$

and a permutation of $G$ leaving $x_1$ unaltered leaves unaltered every $x_i$. It necessarily is identity, whence $G$ is regular of order[4]

$$r = p - 1.$$

[1] The same can be proved for any cyclotomic equation.
[2] By proposition (123).
[3] By proposition (60).
[4] Cf. §§72 and 77. It follows by proposition (124) that a cyclotomic equation is normal.

By proposition (107) the relation

$$x_i = x_1{}^i$$

must remain true if operated on by the permutations of $G$. But a permutation of $G$ converting

$$x_1 \to x_i = x_1{}^i$$

converts

$$x_1{}^i \to x_i{}^i = x_1{}^{i^2}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and is modulo $p$

$$s = (x_1 x_1{}^i x_1{}^{i^2} \ . \ . \ . \ ) \ . \ . \ .$$

If we now take as $i$ a primitive root of $p$:

$$i = g,$$

then one cycle of $s$ contains all $x_i$; it is modulo $p$

$$s = (\epsilon\epsilon^g\epsilon^{g^2} \ . \ . \ . \ ) = (x_1 x_g x_{g^2} \ . \ . \ . \ )$$

of order $p - 1$, whence

$$G = \{s\}.$$

This means that

(147) **every cyclotomic equation for prime roots of unity is cyclic**

and implies that every cyclotomic equation for prime roots of unity is solvable.[1] Besides it is of degree one less than the degree of the resolvent for which we need its roots.[2]

Hence prime roots of unity can be computed by algebraic operations, and we can state the main result of our study in the final form:[3]

(148) **Whenever the group of an equation has a series of subgroups each normal and of prime index in the preceding, the series beginning with the group and ending with identity, then we can solve the equation by algebraic operations which beside the rational operations include the extraction of roots. But when these conditions do not hold, then it is not possible to solve the equation by algebraic operations on numbers which are rationally known.**

[1] Cf. proposition (76).

[2] Note that for a cubic resolvent the cyclotomic equation is of degree two, that is one less. This is why we can solve the general cubic.

[3] Compare §76, also §§40 and 69.

Thus not every algebraic equation has algebraic roots, and the fundamental theorem of algebra is not a theorem of algebra at all.

## §83. DISCRIMINANT OF CYCLOTOMIC EQUATION

The discriminant $\Delta$ of the cyclotomic equation

$$c(x) = x^{p-1} + x^{p-2} + \ldots + x + 1 = 0$$

is determined by the relation[1]

$$(-1)^{\frac{p-1}{2}} \Delta = c'(x_1)c'(x_2) \ldots c'(x_{p-1})$$

since

$$(-1)^{\frac{p-1}{2}} = (-1)^{\frac{(p-1)(p-2)}{2}}$$

$p - 1$ being even and $p - 2$ odd.

Differentiating

$$x^p - 1 = (x - 1)c(x)$$

we obtain

$$px^{p-1} = c(x) + (x - 1)c'(x),$$

and setting

$$x = x_i$$

with $x_i$ standing for any root of the cyclotomic equation, we have

$$px_i^{p-1} = (x_i - 1)c'(x_i).$$

Multiplying such relations for all $x_i$, we find that

$$p^{p-1} = \prod_{i=1}^{p-1} c'(x_i) \prod_{i=1}^{p-1} (1 - x_i) = (-1)^{\frac{p-1}{2}} \Delta \,.\, c(1)$$

since

$$\prod x_i = 1$$

and $p - 1$ is even. But

$$c(1) = p,$$

whence

(149) **the discriminant $\Delta$ of a cyclotomic equation for prime $p$-th roots of unity has the value**

$$\boxed{\Delta = (-1)^{\frac{p-1}{2}} p^{p-2}}.$$

It follows that

$$\sqrt{\Delta} = \pm p^{\frac{p-3}{2}} i^{\frac{p-1}{2}} \sqrt{p}$$

[1] Cf. §27.

with rational $p^{(p-3)/2}$ is real if

$$p \equiv 1 (\text{mod } 4)$$

and imaginary if

$$p \equiv 3 (\text{mod } 4).$$

## §84. APPLICATIONS

It is in the sense of Galois to use cyclic resolvents, but often convenient to solve cyclotomic equations by more elementary methods.

In the computations which follow we shall consider that

$$\epsilon^k = \cos \frac{2k\pi}{p} + i \sin \frac{2k\pi}{p}$$

whenever difficulties arise concerning the sign.

To find primitive roots of $p$ that we shall need, there is no general method but trying.

Examples:

(*a*) For

$$x^4 + x^3 + x^2 + x + 1 = 0$$

and

$$p = 5$$

we can take

$$g \equiv 2 (\text{mod } 5)$$

giving

$$g^2 \equiv 4,\ g^3 \equiv 3,\ g^4 \equiv 1.$$

Hence

$$s = (\epsilon\epsilon^2\epsilon^4\epsilon^3) = (1243)$$

and

$$\begin{aligned} G_4 &= 1, (1243), (14)(23), (1342) \\ G_2 &= 1, (14)(23) \\ G_1 &= 1. \end{aligned}$$

To $G_2$ belongs the function

$$\varphi_1 = x_1 + x_4$$

with the conjugate value

$$\varphi_2 = x_2 + x_3.$$

We have

$$\varphi_1 + \varphi_2 = x_1 + x_2 + x_3 + x_4 = -1,$$

and also

$$\varphi_1\varphi_2 = x_1 + x_2 + x_3 + x_4 = -1$$

since

$$x_i x_k = \epsilon^{i+k (\text{mod } 5)} = x_{i+k (\text{mod } 5)},$$

so that the $\varphi_i$ are roots of the quadratic resolvent

$$\varphi^2 + \varphi - 1 = 0$$

giving

$$\varphi_1 = \frac{-1 + \sqrt{5}}{2} \qquad [\varphi_1 = 2 \cos \frac{2\pi}{5}$$

$$\varphi_2 = \frac{-1 - \sqrt{5}}{2}. \qquad [\varphi_2 = -2 \cos \frac{\pi}{5}$$

To $G_1$ belongs the function

$$\psi_1 = x_1$$

with the conjugate value

$$\psi_2 = x_4.$$

But

$$\psi_1 + \psi_2 = \varphi_1$$
$$\psi_1\psi_2 = \epsilon^0 = 1,$$

so that the $\psi_i$ are roots of the quadratic resolvent

$$\psi^2 + \frac{1 - \sqrt{5}}{2}\psi + 1 = 0$$

giving

$$\psi_1 = x_1 = \frac{-1 + \sqrt{5} + \sqrt{-10 - 2\sqrt{5}}}{4} \qquad [\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$$

$$\psi_2 = x_4 = \frac{-1 + \sqrt{5} - \sqrt{-10 - 2\sqrt{5}}}{4}. \qquad [\cos \frac{2\pi}{5} - i \sin \frac{2\pi}{5}$$

Likewise

$$x_2 = \frac{-1 - \sqrt{5} + \sqrt{-10 + 2\sqrt{5}}}{4} \qquad [-\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}$$

$$x_3 = \frac{-1 - \sqrt{5} - \sqrt{-10 + 2\sqrt{5}}}{4}. \qquad [-\cos \frac{\pi}{5} - i \sin \frac{\pi}{5}$$

---

(*b*) For

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$$

and

$$p = 7$$

we can take

$$g \equiv 3 (\mathrm{mod}\ 7)$$

giving

$$g^2 \equiv 2,\ g^3 \equiv 6,\ g^4 \equiv 4,\ g^5 \equiv 5,\ g^6 \equiv 1.$$

Hence

$$s = (132645)$$

and

$$G_6 = 1,\ (132645),\ (124)(365),\ (16)(34)(25),\ (142)(356),\ (154623)$$
$$G_3 = 1,\ (124)(365),\ (142)(356)$$
$$G_1 = 1.$$

To $G_3$ belongs the function

$$\varphi_1 = x_1 + x_2 + x_4$$

with the conjugate value

$$\varphi_2 = x_3 + x_6 + x_5$$

and

$$\varphi_1 + \varphi_2 = -1$$
$$\varphi_1\varphi_2 = 2.$$

From the quadratic resolvent

$$\varphi^2 + \varphi + 2 = 0$$

we obtain

$$\varphi_1 = \frac{-1 + \sqrt{-7}}{2} \quad [\sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} - \sin\frac{\pi}{7} > 0$$

$$\varphi_2 = \frac{-1 - \sqrt{-7}}{2}. \quad [\sin\frac{\pi}{7} - \sin\frac{2\pi}{7} - \sin\frac{3\pi}{7} < 0$$

To $G_1$ belongs the function

$$\psi_1 = x_1$$

with the conjugate values

$$\psi_2 = x_2$$
$$\psi_3 = x_4,$$

and these functions are roots of the cubic resolvent

$$\psi^3 - \varphi_1\psi^2 + \varphi_2\psi - 1 = 0$$

which we know how to solve.

---

(*c*) Historic interest is attached to the cyclotomic equation

$$x^{16} + x^{15} + \ . \ . \ . \ + x + 1 = 0$$

with

$$p = 17,$$

because here Gauss discovered that the construction of a regular polygon with rule and compass is possible whenever the roots of unity which give its vertices can be computed by quadratic resolvents.

Taking
$$g \equiv 3 (\bmod 17)$$
we obtain
$$s = (1\ 3\ 9\ 10\ 13\ 5\ 15\ 11\ 16\ 14\ 8\ 7\ 4\ 12\ 2\ 6)$$
and
$$G_{16} = \{s\},\ G_8 = \{s^2\},\ G_4 = \{s^4\},\ G_2 = \{s^8\},\ G_1 = 1.$$
Setting
$$\varphi_1 = x_1 + x_9 + x_{13} + x_{15} + x_{16} + x_8 + x_4 + x_2$$
$$\varphi_2 = x_3 + x_{10} + x_5 + x_{11} + x_{14} + x_7 + x_{12} + x_6$$
we have
$$\varphi_1 + \varphi_2 = -1$$
$$\varphi_1\varphi_2 = -4.$$
From the quadratic resolvent
$$\varphi^2 + \varphi - 4 = 0$$
we find
$$\varphi_1 = \frac{-1 + \sqrt{17}}{2}$$
$$\varphi_2 = \frac{-1 - \sqrt{17}}{2}.$$
Setting
$$\chi_1 = x_1 + x_{13} + x_{16} + x_4$$
$$\chi_2 = x_9 + x_{15} + x_8 + x_2$$
$$\chi_3 = x_3 + x_5 + x_{14} + x_{12}$$
$$\chi_4 = x_{10} + x_{11} + x_7 + x_6$$
we have
$$\chi_1 + \chi_2 = \varphi_1$$
$$\chi_1\chi_2 = -1$$
and
$$\chi_3 + \chi_4 = \varphi_2$$
$$\chi_3\chi_4 = -1.$$
From the quadratic resolvent
$$\chi^2 + \frac{1 - \sqrt{17}}{2}\chi - 1 = 0$$
we find
$$\chi_1 = \frac{-1 + \sqrt{17}}{4} + \sqrt{\frac{17 - \sqrt{17}}{8}}$$
$$\chi_2 = \frac{-1 + \sqrt{17}}{4} - \sqrt{\frac{17 - \sqrt{17}}{8}},$$

and from the quadratic resolvent

$$\chi^2 + \frac{1 + \sqrt{17}}{2}\chi - 1 = 0$$

we find

$$\chi_3 = \frac{-1 - \sqrt{17}}{4} + \sqrt{\frac{17 + \sqrt{17}}{8}}$$

$$\chi_4 = \frac{-1 - \sqrt{17}}{4} - \sqrt{\frac{17 + \sqrt{17}}{8}}.$$

Setting

$$\psi_1 = x_1 + x_{16}$$

$$\psi_2 = x_{13} + x_4$$

we have

$$\psi_1 + \psi_2 = \chi_1$$

$$\psi_1\psi_2 = \chi_3,$$

and from the quadratic resolvent

$$\psi^2 - \chi_1\psi + \chi_3 = 0$$

we find $\psi_1$ and $\psi_2$. Finally we set

$$\omega_1 = x_1$$

$$\omega_2 = x_{16}$$

where

$$\omega_1 + \omega_2 = \psi_1$$

$$\omega_1\omega_2 = 1,$$

and from the quadratic resolvent

$$\omega^2 - \psi_1\omega + 1 = 0$$

we find $x_1$ and $x_{16}$.

Since the roots of a cyclotomic equation can be constructed geometrically with rule and compass whenever they can be computed algebraically by square roots, this might seem possible whenever

$$p - 1 = 2^m$$

and

$$p = 2^m + 1,$$

for a cyclic group of order $p - 1$ has a subgroup of any order which is a divisor of $p - 1$, as we know.[1]

But when

$$m = kq$$

[1] Cf. proposition (75).

where $q$ is odd, then $2^m + 1$ is not prime because

$$\frac{(2^k)^q + 1}{2^k + 1} = (2^k)^{q-1} - (2^k)^{q-2} + \ . \ . \ . + 1$$

is integral. Hence it is necessary that

$$m = 2^n$$

and

$$p = 2^{2^n} + 1.$$

Such numbers $p$ are called **Fermat numbers,** and it was believed that they all are prime until Euler[1] discovered that $2^{2^5} + 1$ is divisible by 641.[2]

## §85. METACYCLIC EQUATION

An equation whose group is metacyclic we call a **metacyclic equation.**[3] We do not have to qualify such an equation as irreducible because a metacyclic group is transitive.[4]

When the degree of a metacyclic equation is prime, the equation is solvable by proposition (86); and so is any equation of prime degree whose group is a transitive subgroup of the metacyclic. Conversely, when an irreducible equation of prime degree is solvable, its group is by proposition (86) the metacyclic group or one of its transitive subgroups.

This gives the **Theorem of Galois:**

(150) **If an irreducible equation of prime degree is solvable, its group is the metacyclic group or a transitive subgroup of the metacyclic group.**

When we reduce the group of a metacyclic equation of prime degree along its series of composition, the equation remains irreducible until the last by proposition (84). Only when we reach identity, the equation splits and splits at once into linear factors yielding the roots. The same is true for any equation of prime degree whose group is a transitive subgroup of the metacyclic, is therefore true for any irreducible equation of prime degree which is solvable.

[1] Euler lived 1707–1783.

[2] Now Fermat numbers are known not to be prime for

$$n = 5, 6, 7, 8, 9, 11, 12, 23.$$

[3] Compare footnotes to §§55 and 58.

[4] Cf. proposition (123).

Such equations have a peculiar property:

(151) **All roots of an irreducible but solvable equation of prime degree are rationally expressible in terms of any two of them.**

For the identical permutation is the only permutation of the metacyclic group leaving two roots $x_i$ and $x_k$ unaltered, which is readily verified from the nature of such permutations:

$$t = \binom{z}{\mu z + \nu}.$$

Hence the function

$$v = \alpha_i x_i + \alpha_k x_k$$

belongs to the group identity, and every root of the equation is rationally expressible in terms of $v$, which is to say in terms of $x_i$ and $x_k$.

Conversely:

(152) **An irreducible equation of prime degree whose roots are rationally expressible in terms of any two of them is solvable.**

For the group $G$ of the equation is transitive on the roots

$$x_0, x_1, \ldots, x_{p-1}$$

of the equation,[1] whence its order is a multiple of $p$.[2] But $p$ is prime; by the Theorem of Cauchy the group $G$ has a subgroup of order $p$, and by proposition (25) this subgroup is circular.

Let the subgroup be $C$ formed by the powers of

$$s = (01 \ldots p - 1),$$

where 0 and 1 designate any two roots of the proposition. Unless the subgroup $C$ is normal in the group $G$, there is in $G$ a permutation

$$t = (i_0 i_1 \ldots i_{p-1})$$

similar to $s$ yet not a power of $s$, by proposition (30).

If $\mu$ and $\nu$ are two numbers not congruent modulo $p$, we apply to the relation

$$x_{i_\alpha} = R(x_{i_0}, x_{i_1})$$

[1] Cf. proposition (123).

[2] Cf. proposition (60).

of the proposition the permutations $t^{\mu}s^{-i_{\mu}}$ and $t^{\nu}s^{-i_{\nu}}$ of $G$ and obtain the two relations

$$x_{i_{\alpha+\mu}-i_{\mu}} = R(x_0, x_{i_{1+\mu}-i_{\mu}})$$
$$x_{i_{\alpha+\nu}-i_{\nu}} = R(x_0, x_{i_{1+\nu}-i_{\nu}}),$$

which are true by proposition (107).

The numbers $\mu$ and $\nu$ can be chosen so that

$$i_{\mu+1} - i_{\mu} \equiv i_{\nu+1} - i_{\nu} \equiv k (\text{mod } p),$$

for there are $p$ such differences and they can take only $p - 1$ values incongruent modulo $p$, zero not being possible. This gives

$$x_{i_{\alpha+\mu}-i_{\mu}} = R(x_0, x_k)$$
$$x_{i_{\alpha+\nu}-i_{\nu}} = R(x_0, x_k);$$

whence

$$i_{\alpha+\mu} - i_{\mu} = i_{\alpha+\nu} - i_{\nu}$$

for

$$\alpha = 0, 1, \ldots, p - 1.$$

If these relations are written out in full, it is readily seen that they imply the other:

$$i_{\alpha+1} - i_{\alpha} = \text{constant},$$

say $m$. But this means that

$$t = s^m$$

contrary to our assumption; hence $C$ is normal in $G$. Since the largest group containing $C$ as normal subgroup is metacyclic, the proposition follows.

The roots of an equation being rationally expressible in terms of any two of them thus is a necessary and sufficient condition by which we recognize the equation as solvable when it is irreducible and of prime degree.

It appears that

(153) **an irreducible but solvable equation with real coefficients whose degree is an odd prime has only one real root or real roots alone.**

For it has one real root by a proposition of elementary algebra; and if two roots are real, all are so by proposition (151).

Irreducible but solvable equations of prime degree have another peculiar property:

(154) **An irreducible equation of prime degree is solvable when, and only when, the general resolvent[1] for the metacyclic group having no double roots has one rational root.**

Such a resolvent for the general equation is

$$r(\psi) = (\psi - \psi_1)(\psi - \psi_2) \ . \ . \ . = 0,$$

where $\psi_1$ is a function belonging to the metacyclic group. Its degree is equal to the index of the metacyclic group in the symmetric, which is

$$j_m = 1 \,.\, 2 \ . \ . \ . \ (p - 2);$$

and it is irreducible with coefficients that are symmetric by proposition (119).

Suppose now we substitute for the roots $x_i$ of the general equation those of the special equation given by the proposition. Let no two $\psi_i$ become alike, which implies that $\psi_1$ still belongs to the metacyclic group; this can always be arranged by a proper choice of $\psi_1$.

If the group of the equation is soluble, the root $\psi_1$ of the resolvent has a rational value by property (2) of the group, because $\psi_1$ is unaltered by the largest possible such group, the metacyclic.

Conversely, if the root $\psi_1$ of the resolvent has a rational value, the group of the equation is contained in the metacyclic by property (1), because $\psi_1$ is unaltered by the metacyclic group. Hence the proposition follows.

To the irreducible but solvable equations of prime degree belong the binomial equations. For

(155) **a binomial equation of prime degree in $\Omega$ with no root rational in $\Omega$ is irreducible there but solvable.**

Let such an equation be

$$x^p - c = 0;$$

it has the roots

$$x_0, \epsilon x_0, \ . \ . \ . \ , \epsilon^{p-1}x_0$$

when $x_0$ is one of them and $\epsilon$ a primitive root of unity.

Suppose that

$$x^p - c$$

[1] We mean the resolvent for the general equation with the roots of the given equation substituted.

has a factor

$$x^q + \ldots + d$$

in $\Omega$ with

$$q < p.$$

The product of its roots, which is to say the constant term

$$d = \epsilon^z x_0{}^q$$

with unknown $z$, is rational in $\Omega$.

By proposition (3) which applies also to integers,[1] there exist two integers $a$ and $b$ such that

$$aq - bp = 1.$$

Hence rational in $\Omega$ is also

$$(\epsilon^z x_0{}^q)^a = \epsilon^{az} x_0{}^{bp+1} = \epsilon^{az} x_0 \cdot c^b,$$

where $\epsilon^{az} x_0$ is a root of the equation.

With $c^b$ also $\epsilon^{az} x_0$ then is rational in $\Omega$, which is contrary to the assumption of our proposition.

Writing the roots of the equation in the form

$$x_0, x_1, \ldots, x_{p-1},$$

we have

$$\epsilon = \frac{x_1}{x_0}$$

and

$$x_i = \left(\frac{x_1}{x_0}\right)^i x_0.$$

It appears that all roots of the equation are rationally expressible in terms of two of them, which completes the proof.

As function belonging to the cyclic group

$$C = \{s\}$$

with

$$s = (01 \ldots p - 1)$$

we can take

$$\epsilon = \frac{x_1}{x_0} = \frac{x_2}{x_1} = \ldots = \frac{x_p}{x_{p-1}}.$$

But $\epsilon$ satisfies the cyclotomic equation

$$x^{p-1} + \ldots + x + 1 = 0$$

of degree $p - 1$. In the domain of rational numbers this equation is irreducible by proposition (146) and serves as resolvent

[1] Compare proposition (91).

of the binomial equation, whence the group of the binomial equation is metacyclic containing $C$ as subgroup of index $p - 1$. We conclude that

(156) **a binomial equation of prime degree with no rational root is metacyclic in the domain of rational numbers.**

## §86. QUINTIC EQUATION

The general quintic has a resolvent of degree six, as already Lagrange knew; for the metacyclic group is of index six in the symmetric. And when the resolvent for the metacyclic group takes for a special quintic one rational root, the quintic is solvable by proposition (154). It is possible to construct such a resolvent sextic, but to compute its coefficients is very painful. Somewhat lighter is the task when the quintic is reduced to the **Bring-Jerrard normal form**

$$x^5 + px + q = 0.$$

To reduce the quintic, we first prove that

(157) **every quadratic form is expressible as sum of squares of linear forms.**

By a **form** we mean a homogeneous function, by a quadratic form

$$\varphi = \varphi(x_i)_1^n$$

a homogeneous function of degree two in the $x_i$. If it contains terms in $x_1^2$, we can set

$$\varphi = c^2x_1^2 + 2c\psi x_1 + \omega,$$

where $\psi$ is linear and $\omega$ quadratic in the $x_i$ other than $x_1$. But this is the same as

$$\begin{aligned}\varphi &= (cx_1 + \psi)^2 + (\omega - \psi^2)\\ &= u^2 + (\omega - \psi^2),\end{aligned}$$

where $\omega - \psi^2$ is a quadratic form in the $x_i$ other than $x_1$.

If $\varphi$ contains no square of an $x_i$, we can set

$$\varphi = c^2x_1x_2 + c\chi x_1 + c\psi x_2 + \omega,$$

where $\chi$ and $\psi$ are linear and $\omega$ is quadratic in the $x_i$ other than $x_1$ and $x_2$. But this is the same as

$$\begin{aligned}\varphi = \tfrac{1}{4}[c(x_1 + x_2) + (\chi + \psi)]^2 - \tfrac{1}{4}[c(x_1 - x_2) - (\chi - \psi)]^2\\ + (\omega - \chi\psi) = \tfrac{1}{4}u^2 - \tfrac{1}{4}v^2 + (\omega - \chi\psi),\end{aligned}$$

where $\omega - \chi\psi$ is a quadratic form in the $x_i$ other than $x_1$ and $x_2$.

The quadratic forms obtained in fewer than $n$ letters $x_i$ we then treat as we treated $\varphi$ and continue until $\varphi$ is expressed as sum of squared linear forms. Evidently there can be no more than $n$ such squares.

Jerrard proved, what Bring achieved much earlier for a quintic equation, that

(158) **by a Tschirnhaus transformation involving square and cube roots the second, third and fourth terms of an equation can be removed.**

The general equation

$$x^n + a_1x^{n-1} + \dots + a_n = 0$$

is transformed into

$$y^n + b_1y^{n-1} + \dots + b_n = 0$$

by the substitution

$$y = \alpha_0 + \alpha_1x + \alpha_2x^2 + \alpha_3x^3 + \alpha_4x^4,$$

which is a case of the **Tschirnhaus transformation**

$$y = \frac{\varphi(x)}{\psi(x)}.$$

For this substitution we have

$$y^2 = \beta_0 + \beta_1x + \dots + \beta_{n-1}x^{n-1}$$

$$\cdots\cdots$$

$$y^n = \delta_0 + \delta_1x + \dots + \delta_{n-1}x^{n-1},$$

since powers of $x$ higher than $n - 1$ can be reduced from[1]

$$x^n = -a_1x^{n-1} - \dots - a_n.$$

Any $\beta_i$ is homogeneous of degree two, any $\delta_i$ is homogeneous of degree $n$ in the $\alpha_i$.

The relations between $x$ and $y$ hold for any $x_i$ of the general equation, for every $x_i$ we have a $y_i$, and setting

$$s_i = s(x^i)$$
$$\sigma_i = s(y^i)$$

in the notation of §22 with the subscripts of $x$ and $y$ not marked, we find

$$\sigma_1 = n\alpha_0 + \alpha_1s_1 + \alpha_2s_2 + \alpha_3s_3 + \alpha_4s_4$$
$$\sigma_2 = n\beta_0 + \beta_1s_1 + \dots + \beta_{n-1}s_{n-1}$$

$$\cdots\cdots$$

[1] Compare the same for $\psi$ in §35.

The $s_i$ are computable from the $a_i$, by proposition (34); likewise the $\sigma_i$ are computable from the $b_i$:

$$\begin{aligned}\sigma_1 &= -b_1\\ \sigma_2 &= b_1^2 - 2b_2\\ &\cdots\cdots\end{aligned}$$

Any $\sigma_i$ being homogeneous of degree $i$ in the $\alpha_i$, the same is true for any $b_i$.

From the equation

$$b_1 = 0$$

linear in the $\alpha_i$ we obtain one $\alpha_i$ in terms of the other four and eliminate it from the equations

$$\begin{aligned}b_2 &= 0\\ b_3 &= 0,\end{aligned}$$

putting the equation for $b_2$ by proposition (157) in the form

$$\lambda_1 u_1^2 + \lambda_2 u_2^2 + \lambda_3 u_3^2 + \lambda_4 u_4^2 = 0,$$

where any $u_i$ is linear in the $\alpha_i$.

Then we find such $\alpha_i$ that

$$\begin{aligned}\lambda_1 u_1^2 + \lambda_2 u_2^2 &= 0\\ \lambda_3 u_3^2 + \lambda_4 u_4^2 &= 0,\end{aligned}$$

or

$$\begin{aligned}\sqrt{\lambda_1}\, u_1 &= \sqrt{-\lambda_2}\, u_2\\ \sqrt{\lambda_3}\, u_3 &= \sqrt{-\lambda_4}\, u_4,\end{aligned}$$

which is to say express two $\alpha_i$ in terms of the remaining two.

Substituting into the equation

$$b_3 = 0,$$

we have an equation homogeneous of degree three in two $\alpha_i$. Choosing one $\alpha_i$ at will, we solve for the other, and all $\alpha_i$ are determined.

With the $\alpha_i$ the substitution is determined which removes from the general equation the terms in $y^{n-1}$, $y^{n-2}$, $y^{n-3}$.

Thus we can take the general quintic

$$x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0$$

to be

$$x^5 + px + q = 0,$$

with

$$\begin{aligned}a_1 = a_2 = a_3 &= 0\\ a_4 = p,\ a_5 &= q.\end{aligned}$$

The function $\psi$ of its roots given in §58 and belonging to the half-metacyclic group takes under the alternating the conjugate values:[1]

$$\begin{aligned}
\psi_1 &= x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1\\
\psi_2 &= x_2x_1 + x_1x_4 + x_4x_3 + x_3x_5 + x_5x_2\\
\psi_3 &= x_2x_4 + x_4x_5 + x_5x_3 + x_3x_1 + x_1x_2\\
\psi_4 &= x_5x_4 + x_4x_1 + x_1x_3 + x_3x_2 + x_2x_5\\
\psi_5 &= x_2x_4 + x_4x_1 + x_1x_5 + x_5x_3 + x_3x_2\\
\psi_6 &= x_2x_5 + x_5x_1 + x_1x_3 + x_3x_4 + x_4x_2.
\end{aligned}$$

The function $\psi'$ conjugate to $\psi$ in the metacyclic group takes under the alternating the corresponding values:

$$\begin{aligned}
\psi_1' &= x_1x_3 + x_1x_4 + x_2x_4 + x_2x_5 + x_3x_5\\
\psi_2' &= x_1x_3 + x_1x_5 + x_2x_3 + x_2x_4 + x_4x_5\\
\psi_3' &= x_1x_4 + x_1x_5 + x_2x_3 + x_2x_5 + x_3x_4\\
\psi_4' &= x_1x_2 + x_1x_5 + x_2x_4 + x_3x_4 + x_3x_5\\
\psi_5' &= x_1x_2 + x_1x_3 + x_2x_5 + x_3x_4 + x_4x_5\\
\psi_6' &= x_1x_2 + x_1x_4 + x_2x_3 + x_3x_5 + x_4x_5
\end{aligned}$$

such that

$$\psi_i + \psi_i' = a_2.$$

The function

$$\omega_1 = \psi_1 - \psi_1'$$

belongs to the half-metacyclic group; the six conjugate values of $\omega_1$ under the alternating group are roots of the resolvent

$$(\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3)(\omega - \omega_4)(\omega - \omega_5)(\omega - \omega_6) = 0$$

whose coefficients belong to the alternating group, by proposition (50). Since the $\omega_i$ change their sign when the root of the discriminant does, the resolvent is:

$$\omega^6 + \lambda_2\omega^4 + \lambda_4\omega^2 + \lambda_6 - \sqrt{\Delta}(\lambda_1\omega^5 + \lambda_3\omega^3 + \lambda_5\omega) = 0,$$

where the $\lambda_i$ are symmetric in the $x_i$.

We find for

$$\lambda_1\sqrt{\Delta},\ \lambda_2,\ \lambda_3\sqrt{\Delta},\ \lambda_4,\ \lambda_5\sqrt{\Delta},\ \lambda_6$$

the total degrees

$$2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12$$

in the $x_i$. But $\sqrt{\Delta}$ is of total degree ten, whence $\lambda_5$ is some number and

$$\lambda_1 = \lambda_3 = 0.$$

[1] We apply to $\psi_1$ the permutations: (12)(34), (1243)(15) = (12435) (1243)(25) = (15243), (1243)(35) = (12453), (1243)(45) = (12543).

The weight for

$$\lambda_2, \lambda_4, \lambda_6$$

in the $a_i$ is

$$4 \quad 8 \quad 12,$$

by §24. Therefore $a_5$ is not contained in $\lambda_2$; it is not contained in $\lambda_4$, $\lambda_6$ either because it could be there only in combination with $a_1$, $a_2$, $a_3$ which are zero.

Consequently $\lambda_2$, $\lambda_4$, $\lambda_6$ are not affected if we assume that

$$q = 0.$$

Then

$$x_1 = 0,\ x_2 = (-p)^{1/4},\ x_3 = i(-p)^{1/4}$$
$$x_4 = -(-p)^{1/4},\ x_5 = -i(-p)^{1/4},$$

whence

$$\sqrt{\Delta} = \prod_{i<k}(x_i - x_k) = -16\sqrt{p^5}$$
$$\Delta = 256\, p^5.$$

From

$$\psi_i + \psi_i' = 0$$
$$\psi_i - \psi_i' = \omega_i$$

we have

$$\omega_i = 2\psi_i,$$

and substituting the values for the $x_i$:

$$\omega_1 = \omega_3 = \omega_4 = \omega_5 = -2\sqrt{p}$$
$$\omega_2 = (4 + 2i)\sqrt{p},\ \omega_6 = (4 - 2i)\sqrt{p}.$$

The resolvent for this case is

$$(\omega + 2\sqrt{p})^4(\omega^2 - 8\omega\sqrt{p} + 20p)$$
$$= \omega^6 - 20\, p\omega^4 + 240\, p^2\omega^2 + 512\sqrt{p^5}\omega + 320\, p^3 = 0.$$

Hence the resolvent is always

$$\omega^6 - 20\, p\omega^4 + 240\, p^2\omega^2 - 32\sqrt{\Delta}\,\omega + 320\, p^3 = 0,$$

considering that $\lambda_5$ does not depend on $q$.

The discriminant is expressible in terms of $p$ and $q$. Being of total degree twenty in the $x_i$, it is

$$\Delta = kp^5 + lq^4.$$

We find $k$ by setting $q = 0$ and $l$ by setting $p = 0$ and then have

$$\Delta = 2^8p^5 + 5^5q^4.$$

In terms of $\psi$ the resolvent is

$$\psi^6 - 5p\psi^4 + 15p^2\psi^2 - \sqrt{\Delta}\,\psi + 5p^3 = 0.$$

Substituting

$$\varphi_i = \psi_i^2$$

we find:

(159) **The Bring-Jerrard normal form**

$$\boxed{x^5 + px + q = 0}$$

**of the general quintic has a resolvent sextic**

$$\boxed{(\varphi^3 - 5p\varphi^2 + 15p^2\varphi + 5p^3)^2 = \Delta\varphi}$$

**for the metacyclic group. It is the resolvent also of a special quintic if the special quintic is irreducible.**

For the degree of the resolvent is six, its coefficients are symmetric and its root $\varphi_1$ belongs to the metacyclic group.

When the roots $x_i$ of a special quintic are substituted, $\varphi_1$ still belongs to the metacyclic group if the quintic is irreducible, because then no $\varphi_i$ are alike.

The resolvent in $\varphi$ has equal roots only when the resolvent in $\psi$ has equal roots or roots differing by the sign alone.

Two roots $\psi_i$ differing by the sign alone are possible only when the resolvent in $\psi$ has

$$\Delta = 0,$$

which is to say when the quintic has double roots. But this case is excluded.

Equal roots $\psi_i$ are possible only when the resolvent in $\psi$ has a root in common with

$$6\psi^5 - 20p\psi^3 + 30p^2\psi - \sqrt{\Delta} = 0,$$

or eliminating $\sqrt{\Delta}$:

$$5\psi^6 - 15p\psi^4 + 15p^2\psi^2 - 5p^3 = 5(\psi^2 - p)^3 = 0,$$

which is to say when

$$\varphi = p.$$

But this means that

$$q = 0,$$

which is readily verified from the resolvent in $\varphi$ with the value of $\Delta$ put in. In this case the quintic is reducible, which proves the proposition.

Hence it follows that any quintic in the Bring-Jerrard normal form is solvable when, and only when, it is irreducible and the general resolvent for the metacyclic group has a rational root.

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